CSE 548: Algorithms

Randomized Algorithms

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Example 1: Routing

What is the best way to route a packet from $X$ to $Y$, esp. in high speed, high volume networks

A: Pick the shortest path from $X$ to $Y$

B: Send the packet to a random node $Z$, and let $Z$ route it to $Y$ (possibly using a shortest path from $Z$ to $Y$)
Example 1: Routing

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- Valiant showed in 1981 that surprisingly, B works better!
  
  - Turing award recipient in 2010
Example 2: Transmitting on shared network

- What is the best way for $n$ hosts to share a common network?
  
  A: Give each host a turn to transmit
  
  B: Maintain a queue of hosts that have something to transmit, and use a FIFO algorithm to grant access
  
  C: Let every one try to transmit. If there is contention, use random choice to resolve it.

- Which choice is better?
Topics

1. Intro
2. Probability Basics
   - Discrete Probability
   - Coupon Collection
   - Birthday
   - Balls and Bins
3. Taming distribution
   - Quicksort
4. Probabilistic Algorithms
   - Caching
   - Hashing
   - Universal/Perfect hash
   - Bloom filter
   - Rabin-Karp
   - Prime testing
   - Min-cut
Randomization can often:

- Enable the use of a simpler algorithm
- Cut down the amount of book-keeping
- Support decentralized decision-making
- Ensure fairness

**Examples:**

**Media access protocol:** Avoids need for coordination — important here, because coordination needs connectivity!

**Load balancing:** Instead of maintaining centralized information about processor loads, dispatch jobs randomly.

**Congestion avoidance:** Similar to load balancing
Set Theory and Probability

- A countable *sample space* $S$ is a nonempty countable set.
- An *outcome* $\omega$ is an element of $S$.
- A *probability function* $Pr : S \rightarrow \mathbb{R}$ is a total function such that
  - $Pr[\omega] \geq 0$ for all $\omega \in S$, and
  - $\sum_{\omega \in S} Pr[\omega] = 1$
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  - $Pr[\omega] \geq 0$ for all $\omega \in S$, and
  - $\sum_{\omega \in S} Pr[\omega] = 1$
- An event $E$ is a subset of $S$. Its probability is given by:
  $$Pr[E] = \sum_{\omega \in E} Pr[\omega]$$
Probability Rules from Set Theory

Many probability rules follow from the rules on set cardinality

**Sum Rule:** If \( E_0, E_1, \ldots, E_n, \ldots \) are pairwise disjoint events, then

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\Pr[\bigcup_{n \in \mathbb{N}} E_n] = \sum_{n \in \mathbb{N}} \Pr[E_n]
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**Inclusion–Exclusion:**

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$$

**Union Bound:** $Pr[A \cup B] \leq Pr[A] + Pr[B]$
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**Monotonicity:** $A \subseteq B \rightarrow Pr[A] \leq Pr[B]$
Uniform Probability Spaces

A finite probability space $\mathcal{S}$ said to be uniform if $Pr[\omega]$ is the same for all $\omega$. In such spaces:

$$Pr[E] = \frac{|E|}{|\mathcal{S}|}$$

We often this assumption.
Conditional Probability

- Probability of an event under a condition
- The condition limits consideration to a subset of outcomes
  - Consider this subset (rather than whole of $S$) as the space of all possible outcomes

\[
Pr[X | Y] = \frac{Pr[X \cap Y]}{Pr[Y]}
\]
Extending Probability Rules for Conditional Probability

Product Rule 2: \( Pr[E_1 \cap E_2] = Pr[E_1] \cdot Pr[E_2|E_1] \)
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Product Rule 3: \( Pr[E_1 \cap E_2 \cap E_3] = Pr[E_1] \cdot Pr[E_2|E_1] \cdot Pr[E_3|E_1 \cap E_2] \)
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Bayes’ Rule: \( Pr[B|A] = \frac{Pr[A|B] \cdot Pr[B]}{Pr[A]} \)
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Total Probability Law: $Pr[A] = Pr[A|E] \cdot Pr[E] + Pr[A|\bar{E}] \cdot Pr[\bar{E}]$

Total Probability Law 2: If $E_i$ are mutually disjoint and $Pr[\bigcup E_i] = 1$ then
$Pr[A] = \sum Pr[A|E_i] \cdot Pr[E_i]$

Inclusion-Exclusion: $Pr[A \cup B|C] = Pr[A|C] + Pr[B|C] - Pr[A \cap B|C]$
Independence

An event $A$ is independent of $B$ iff the following (equivalent) conditions hold:

- $Pr[A|B] = Pr[A]$ 
- $Pr[A \cap B] = Pr[A] \cdot Pr[B]$ 
- $B$ is independent of $A$

Often, independence is an assumption.

Definition can be generalized to 3 (or $n$) events. Events $E_1$, $E_2$ and $E_3$ a are mutually independent iff all of the following hold:

- $Pr[E_1 \cap E_2] = Pr[E_1] \cdot Pr[E_2]$ 
- $Pr[E_2 \cap E_3] = Pr[E_2] \cdot Pr[E_3]$ 
- $Pr[E_1 \cap E_3] = Pr[E_1] \cdot Pr[E_3]$ 
- $Pr[E_1 \cap E_2 \cap E_3] = Pr[E_1] \cdot Pr[E_2] \cdot Pr[E_3]$
Coupon Collector Problem

- Suppose that your favorite cereal has a coupon inside. There are $n$ types of coupons, but only one of them in each box. How many boxes will you have to buy before you can expect to have all of the $n$ types?

- What is your guess?
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Let us work out the expectation. Let us say that you have so far $j - 1$ types of coupons, and are now looking to get to the $j$th type. Let $X_j$ denote the number of boxes you need to purchase before you get the $j + 1$th type.
Coupon Collector Problem

- Note $E[X_j] = 1/p_j$, where $p_j$ is the probability of getting the $j$th coupon.
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- Note $p_j = (n - j)/n$, so $E[X_j] = n/(n - j)$
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- We have all $n$ types when we finish the $X_{n-1}$ phase:
  \[
  E[X] = \sum_{i=0}^{n-1} E[X_j] = \sum_{i=0}^{n-1} n/(n - j) = nH(n)
  \]
- Note $H(n)$ is the harmonic sum, and is bounded by $\ln n$
Coupon Collector Problem

- Note $E[X_j] = 1/p_j$, where $p_j$ is the probability of getting the $jth$ coupon.

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- We have all $n$ types when we finish the $X_{n-1}$ phase:

$$E[X] = \sum_{i=0}^{n-1} E[X_j] = \sum_{i=0}^{n-1} \frac{n}{n - j} = nH(n)$$

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- Perhaps unintuitively, you need to buy $\ln n$ cereal boxes to obtain one useful coupon.
Birthday Paradox

What is the smallest size group where there are at least two people with the same birthday?

- 365
- 183
- 61
- 25
Birthday Problem

- The probability that two students have different birthdays: \( \frac{364}{365} \)

- In a class of \( n \), there are \( \binom{n}{2} \) pairs of students to consider.
  - If we assume that whether one pair shares a birthday is independent of another, we can simply multiply these probabilities
    \[
    Pr(\text{no two persons with same birthday}) \approx \left( \frac{364}{365} \right)^\binom{n}{2} \approx \left( \frac{364}{365} \right)^{n^2/2}
    \]

- For \( n = 44 \), this formula yields a probability of 7%
  - \( n = 23 \) is enough to have better than even chance of finding two with the same birthday.
Birthday Problem: More Accurate Approach

- What is the probability of finding two people with the same birthday in this class?
- There are $365^n$ possible sequences of birthdays for $n$ people
  - We assume these are all equally likely
- Number of sequences without repetition: $365 \cdot 364 \cdots (365 - (n - 1))$
- Probability that no two of $n$ persons have same birthday:
  \[
  \frac{365}{365} \cdot \frac{365 - 1}{365} \cdots \frac{365 - (n - 1)}{365} = \left( 1 - \frac{0}{365} \right) \left( 1 - \frac{1}{365} \right) \cdots \left( 1 - \frac{n - 1}{365} \right)
  \]
- Use the approximation $(1 - x) < e^{-x}$ to derive an upper bound:
  \[
  Pr(\text{no two persons with same birthday}) < e^0 \cdot e^{-\frac{1}{365}} \cdot e^{-\frac{n-1}{365}} = e^{-\frac{1}{365} \sum_{i=1}^{n-1} i} = e^{\frac{-n(n-1)}{2 \cdot 365}}
  \]
- For $n = 44$, this evaluates to 7.5\%
Birthday Paradox Vs Coupon Collection

- Two sides of the same problem
  
  **Coupon Collection**: What is the minimum number of samples needed to cover every one of $N$ values
  
  **Birthday problem**: What is the maximum number of samples that can avoid covering any value more than once?

So, if we want enough people to ensure that every day of the year is covered as a birthday, we will need $365 \ln 365 \approx 2153$ people!

Almost 100 times as many as needed for one duplicate birthday!
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- What is the maximum number of balls in any bin?
  - Such problems arise in load-balancing, hashing, etc.
Balls and Bins: Max Occupancy

- Probability $p_{1,k}$ that the first bin receives at least $k$ balls:
  - Choose $k$ balls in $\binom{m}{k}$ ways.
  - These $k$ balls should fall into the first bin: prob. is $(1/n)^k$.
  - Other balls may fall anywhere, i.e., probability $1$:\(^1\)

\[
\binom{m}{k} \left( \frac{1}{n} \right)^k = \frac{m \cdot (m-1) \cdots (m-k+1)}{k!n^k} \leq \frac{m^k}{k!n^k}
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\(^1\)This is actually an upper bound, as there can be some double counting.
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  - Let $m = n$, and use Sterling’s approx. $k! \approx \sqrt{2\pi k} (k/e)^k$:
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    P_k = \sum_{i=1}^{n} p_{i,k} \leq n \cdot \frac{1}{k!} \leq n \cdot \left( \frac{e}{k} \right)^k
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    \]
  - Some arithmetic simplification will show that $P_k < 1/n$ when
    \[
    k = \frac{3 \ln n}{\ln \ln n}
    \]

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Balls and Bins: Summary of Results

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- Expected number of empty bins: $ne^{-m/n}$
- Max. balls in any bin when $m = n$:
  \[ \Theta(\ln n / \ln \ln n) \]

- This is a probabilistic bound: chance of finding any bin with higher occupancy is $1/n$ or less.
- Note that the absolute maximum is $n$. 
Randomized Quicksort

- Picks a pivot at random. What is its complexity?

If pivot index is picked uniformly at random over the interval $[l, h]$, then:

- every array element is equally likely to be selected as the pivot
- every partition is equally likely

thus, the expected complexity of randomized quicksort is given by:

$$T(n) = n + \frac{1}{n} \sum_{i=1}^{n-1} (T(i) + T(n-i))$$

Summary: Input need not be random

Expected $O(n \log n)$ performance comes from externally forced randomness in picking the pivot.
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  - But we can’t predict the future!
- Result: many candidates for eviction. How can we avoid making bad (worst-case) choices repeatedly, even if input behaves badly?
- Approach: pick one of the candidates at random!
Hash Tables

- **A data structure for implementing:**
  - **Dictionaries:** Fast look up of a record based on a key.
  - **Sets:** Fast membership check.
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- Support expected $O(1)$ time *lookup, insert, and delete*
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- Two main questions:
  - *How to avoid $O(n)$ worst case behavior?*
  - How to ensure *average case performance* can be realized *for arbitrary distribution of keys?*
Hash Table Implementation

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- Sometimes, we fold the mod operation into $h$. 
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  - Sometimes, we fold the mod operation into $h$.
- Array elements typically called *buckets*
- *Collisions bound to occur* since $s \ll |\mathcal{U}|$
  - Either $h(x) = h(y)$, or
  - $h(x) \neq h(y)$ but $h(x) \equiv h(y) \pmod{s}$
Collisions in Hash tables

- **Load factor** $\alpha$: Ratio of number of keys to number of buckets

If keys were random:

- What is the max $\alpha$ if we want $\leq 1$ collisions in the table?
- If $\alpha = 1$, what is the maximum number of collisions to expect?

Both questions can be answered from balls-and-bins results: $1/\sqrt{n}$, and $O(\ln n / \ln \ln n)$.

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Chained Hash Table

- Each bucket is a linked list.
- Any key that hashes to a bucket is inserted into that bucket.
- What is the average search time, as a function of $\alpha$?
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- Any key that hashes to a bucket is inserted into that bucket.
- What is the average search time, as a function of $\alpha$?
  - It is $1 + \alpha$ if:
    - you assume that the distribution of lookups is independent of the table entries, OR,
    - the chains are not too long (i.e., $\alpha$ is small)
Open addressing

- If there is a collision, probe other empty slots
  - **Linear probing:** If $h(x)$ is occupied, try $h(x) + i$ for $i = 1, 2, ...$
  - **Binary probing:** Try $h(x) \oplus i$, where $\oplus$ stands for exor.
  - **Quadratic probing:** For $i$th probe, use $h(x) + c_1i + c_2i^2$
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- Criteria for secondary probes
  - **Completeness:** Should cycle through all possible slots in table
  - **Clustering:** Probe sequences shouldn’t coalesce to long chains
  - **Locality:** Preserve locality; typically conflicts with clustering.
Open addressing

- If there is a collision, probe other empty slots
  - **Linear probing:** If \( h(x) \) is occupied, try \( h(x) + i \) for \( i = 1, 2, \ldots \)
  - **Binary probing:** Try \( h(x) \oplus i \), where \( \oplus \) stands for exor.
  - **Quadratic probing:** For \( i \)th probe, use \( h(x) + c_1 i + c_2 i^2 \)

- **Criteria for secondary probes**
  - **Completeness:** Should cycle through all possible slots in table
  - **Clustering:** Probe sequences shouldn’t coalesce to long chains
  - **Locality:** Preserve locality; typically conflicts with clustering.

- Average search time can be \( O(1/(1 - \alpha)^2) \) for linear probing, and \( O(1/(1 - \alpha)) \) for quadratic probing.
Chaining Vs Open Addressing

- Chaining leads to fewer collisions
- Clustering causes more collisions w/ open addressing for same $\alpha$
- However, for lean tables, open addressing uses half the space of chaining, so you can use a much lower $\alpha$ for same space usage.
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  - Can be mitigated w/ arrays for buckets instead of linked lists

- Not all quadratic probes cover all slots (but some can)
Resizing

- Hard to predict the right size for hash table in advance
  - Ideally, $0.5 \leq \alpha \leq 1$, so we need an accurate estimate
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  - When $\alpha$ becomes too large (or small), rehash into a bigger (or smaller) table
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  - Exercise: How to ensure amortized $O(1)$ cost when you resize up as well as down?
Average Vs Worst Case

- Worst case search time is $O(n)$ for a table of size $n$
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Average Vs Worst Case

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- *With hash tables, it is all about avoiding the worst case, and achieving the average case*
- Two main challenges:
  - *Input is not random*, e.g., names or IP addresses.
  - Even when input is random, $h$ may cause “lumping,” or non-uniform dispersal of $\mathcal{U}$ to the set $\{1, \ldots, n\}$
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Two main techniques
- Universal hashing
- Perfect hashing
Universal Hashing

- No single hash function can be good on all inputs
  - Any function \( U \rightarrow \{1, \ldots, n\} \) must map \( |U|/n \) inputs to same value!

Note: \( |U| \) can be much, much larger than \( n \).
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Definition

A family of hash functions $\mathcal{H}$ is universal if

$$Pr_{h \in \mathcal{H}}[h(x) = h(y)] = \frac{1}{n} \quad \text{for all} \ x \neq y$$
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**Meaning:** If we pick $h$ at random from the family $\mathcal{H}$, then, probability of collisions is the same for any two elements.

Contrast with non-universal hash functions such as

$$h(x) = ax \mod n, \quad (a \text{ is chosen at random})$$

Note $y$ and $y + kn$ collide with a probability of 1 for every $a$. 


Universal Hashing Using Multiplication

Observation (Multiplication Modulo Prime)

*If $p$ is a prime and $0 < a < p$*

- $\{1a, 2a, 3a, \ldots, (p - 1)a\} = \{1, 2, \ldots, p - 1\} (mod\ p)$

- $\forall a \exists b\ ab \equiv 1\ (mod\ p)$
Observation (Multiplication Modulo Prime)

If $p$ is a prime and $0 < a < p$

- $\{1a, 2a, 3a, \ldots, (p - 1)a\} = \{1, 2, \ldots, p - 1\} \pmod{p}$
- $\forall a \exists b \ ab \equiv 1 \pmod{p}$

Prime multiplicative hashing

Let the key $x \in \mathcal{U}$, $p > |\mathcal{U}|$ be prime, and $0 < r < p$ be random. Then

$$h(x) = (rx \pmod{p}) \pmod{n}$$

is universal.

Prove: $Pr[h(x) = h(y)] = \frac{1}{n}$, for $x \neq y$
Universality of prime multiplicative hashing

- Need to show $Pr[h(x) = h(y)] = \frac{1}{n}$, for $x \neq y$

- $h(x) = h(y)$ means $(rx \mod p) \mod n = (ry \mod p) \mod n$

- Note $a \mod n = b \mod n$ means $a = b + kn$ for some integer $k$. Using this, we eliminate $\mod n$ from above equation to get:

$$r x \mod p = kn + ry \mod p, \text{ where } k \leq \lfloor p/n \rfloor$$

$$r x \equiv kn + ry \pmod{p}$$

$$r(x - y) \equiv kn \pmod{p}$$

$$r \equiv kn(x - y)^{-1} \pmod{p}$$

- So, $x, y$ collide if $r = n(x - y)^{-1}, 2n(x - y)^{-1}, \ldots, \lfloor p/n \rfloor n(x - y)^{-1}$

- In other words, $x$ and $y$ collide for $p/n$ out of $p$ possible values of $r$, i.e., collision probability is $1/n$
Binary multiplicative hashing

- Faster: avoids need for computing modulo prime
Binary multiplicative hashing

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- When $|\mathcal{U}| < 2^w$, $n = 2^l$ and $a$ an odd random number

$$h(x) = \left\lfloor \frac{ax \mod 2^w}{2^{w-l}} \right\rfloor$$
Binary multiplicative hashing

- Faster: avoids need for computing modulo prime
- When $|\mathcal{U}| < 2^w$, $n = 2^l$ and $a$ an odd random number
  \[ h(x) = \left\lfloor \frac{ax \mod 2^w}{2^w - l} \right\rfloor \]
- Can be implemented efficiently if $w$ is the wordsize:
  \[(a^*x) >> (\text{WORDSIZE}-\text{HASHBITS})\]
Faster: avoids need for computing modulo prime

When $|\mathcal{U}| < 2^w$, $n = 2^l$ and $a$ an odd random number

$$h(x) = \left\lfloor \frac{ax \mod 2^w}{2^{w-l}} \right\rfloor$$

Can be implemented efficiently if $w$ is the wordsize:

$$(a \times x) \gg (\text{WORDSIZE-HASHBITS})$$

Scheme is near-universal: collision probability is $O(1)/2^l$
Prime Multiplicative Hash for Vectors

Let $p$ be a prime number, and the key $x$ be a vector $[x_1, \ldots, x_k]$ where $0 \leq x_i < p$. Let 

$$h(x) = \sum_{i=1}^{k} r_ix_i \ (\text{mod } p)$$

If $0 < r_i < p$ are chosen at random, then $h$ is universal.
Prime Multiplicative Hash for Vectors

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$$h(x) = \sum_{i=1}^{k} r_i x_i \pmod{p}$$

If $0 < r_i < p$ are chosen at random, then $h$ is universal.

- Strings can also be handled like vectors, or alternatively, as a polynomial evaluated at a random point $a$, with $p$ a prime:

$$h(x) = \sum_{i=0}^{l} x_i a^i \mod p$$
Universality of multiplicative hashing for vectors

- Since $x \neq y$, there exists an $i$ such that $x_i \neq y_i$
Universality of multiplicative hashing for vectors

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- Since \( y_i, x_i < p \), it is easy to see from this equation that the collision-causing value of \( r_i \) is distinct for distinct \( y_i \).
- Viewed another way, exactly one of \( p \) choices of \( r_i \) would cause a collision between \( x_i \) and \( y_i \), i.e., \( \Pr_h[h(x) = h(y)] = 1/p \)
Perfect hashing

**Static:** Pick a hash function (or set of functions) that avoids collisions for a given set of keys
Perfect hashing

**Static:** Pick a hash function (or set of functions) that avoids collisions for a given set of keys

**Dynamic:** Keys need not be static.

**Approach 1:** Use $O(n^2)$ storage. Expected collision on $n$ items is 0. But too wasteful of storage.

Don’t forget: more memory usually means less performance due to cache effects.
Perfect hashing

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Don’t forget: more memory usually means less performance due to cache effects.

**Approach 2:** Use a secondary hash table for each bucket of size $n_i^2$, where $n_i$ is the number of elements in the bucket.

Uses only $O(n)$ storage, *if h is universal*
Hashing Summary

- Excellent average case performance
- Pointer chasing is expensive on modern hardware, so improvement from $O(\log n)$ of binary trees to expected $O(1)$ for hash tables is significant.
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  - Pointer chasing is expensive on modern hardware, so improvement from $O(\log n)$ of binary trees to expected $O(1)$ for hash tables is significant.

- But all benefits will be reversed if collisions occur too often
  - Universal hashing is a way to ensure expected average case *even when input is not random*.

- Perfect hashing can provide efficient performance even in the worst case, but the benefits are likely small in practice.
Probabilistic Algorithms

- Algorithms that produce the correct answer with some probability

- By re-running the algorithm many times, we can increase the probability to be arbitrarily close to 1.0.
Bloom Filters

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Bloom Filters

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- What if you want to store very large keys?
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What if you want to store very large keys?

**Radical idea:** Don’t store the key in the table!

- Potentially $w$-fold space reduction
Bloom Filters

- To reduce collisions, use multiple hash functions $h_1, \ldots, h_k$
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Images from Wikipedia Commons
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Images from Wikipedia Commons

- Membership check for $y$: all $B[h_i(y)]$ should be set
- No false negatives, but false positives possible
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Bloom Filters: False positives

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Bloom Filters: False positives

- Prob. that a bit is \textit{not} set by \( h_1 \) on inserting a key is \( (1 - 1/m) \)
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- The probability it is not set after \( r \) key inserts is \( (1 - 1/m)^{kr} \approx e^{-kr/m} \)
- Complementing, the prob. \( p \) that a certain bit is set is \( 1 - e^{-kr/m} \)
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- Complementing, the prob. $p$ that a certain bit is set is $1 - e^{-kr/m}$

- For a false positive on a key $y$, all the bits that it hashes to should be a 1. This happens with probability

\[
(1 - e^{-kr/m})^k = (1 - p)^k
\]
Bloom Filters

Consider

\[ (1 - e^{-kr/m})^k \]
Bloom Filters

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- Note that the table can potentially store very large number of entries with very low false positives

- For instance, with \( k = 20 \), \( m = 10^9 \) bits (12M bytes), and a false positive rate of \( 2^{-10} = 10^{-3} \), can store 60M keys of arbitrary size!
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- **Exercise**: What is the optimal value of \( k \) to minimize false positive rate for a given \( m \) and \( r \)?
  - But large \( k \) values introduce high overheads
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Important: Bloom filters can be used as a prefilter, e.g., if actual keys are in secondary storage (e.g., files or internet repositories)
Using arithmetic for substring matching

**Problem:** Given strings $T[1..n]$ and $P[1..m]$, find occurrences of $P$ in $T$ in $O(n + m)$ time.

**Idea:** To simplify presentation, assume $P, T$ range over $[0-9]$

- Interpret $P[1..m]$ as digits of a number
  \[ p = 10^{m-1}P[1] + 10^{m-2}P[2] + \cdots + 10^{m-m}P[m] \]

- Similarly, interpret $T[i..(i + m - 1)]$ as the number $t_i$

- Note: $P$ is a substring of $T$ at $i$ iff $p = t_i$

- To get $t_{i+1}$, shift $T[i]$ out of $t_i$, and shift in $T[i + m]$:
  \[ t_{i+1} = (t_i - 10^{m-1}T[i]) \cdot 10 + T[i + m] \]

We have an $O(n + m)$ algorithm. Almost: we still need to figure out how to operate on $m$-digit numbers in constant time!
Rabin-Karp Fingerprinting

Key Idea

- Instead of working with $m$-digit numbers,
- perform all arithmetic modulo a *random* prime number $q$,
- where $q > m^2$ fits within wordsize

All observations made on previous slide still hold

- Except that $p = t_i$ does not guarantee a match
- Typically, we expect matches to be infrequent, so we can use $O(m)$ exact-matching algorithm to confirm probable matches.
Carter-Wegman-Rabin-Karp Algorithm

Difficulty with Rabin-Karp: Need to generate random primes, which is not an efficient task.

New Idea: Make the radix random, as opposed to the modulus
- We still compute modulo a prime \( q \), but it is not random.

Alternative interpretation: We treat \( P \) as a polynomial

\[
p(x) = \sum_{i=1}^{m} P[m - i] \cdot x^i
\]

and evaluate this polynomial at a randomly chosen value of \( x \)

Like any probabilistic algorithm we can increase correctness probability by repeating the algorithm with different randoms.
- Different prime numbers for Rabin-Karp
- Different values of \( x \) for CWRK
Carter-Wegman-Rabin-Karp Algorithm

\[ p(x) = \sum_{i=1}^{m} P[m - i] \cdot x^i \]

*Random choice does not imply high probability of being right.*

- You need to explicitly establish correctness probability.
Carter-Wegman-Rabin-Karp Algorithm

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So, what is the likelihood of false matches?

- A false match occurs if \( p_1(x) = p_2(x) \), i.e., \( p_1(x) - p_2(x) = p_3(x) = 0 \).
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- Arithmetic modulo prime defines a *field*, so an \( m \)th degree polynomial has \( m + 1 \) roots.
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- Arithmetic modulo prime defines a field, so an \( m \)th degree polynomial has \( m + 1 \) roots.

- Thus, \( (m + 1)/q \) of the \( q \) (recall \( q \) is the prime number used for performing modulo arithmetic) possible choices of \( x \) will result in a false match, i.e., probability of false positive = \( (m + 1)/q \).
Primality Testing

Fermat’s Theorem

\[ a^{p-1} \equiv 1 \pmod{p} \]
Primality Testing

Fermat’s Theorem

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Recall \( \{1a, 2a, 3a, \ldots, (p - 1)a\} \equiv \{1, 2, \ldots, p - 1\} \pmod{p} \)
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**Fermat’s Theorem**

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- Recall \( \{1a, 2a, 3a, \ldots, (p - 1)a\} \equiv \{1, 2, \ldots, p - 1\} \pmod{p} \)

- Multiply all elements of both sides:

\[ (p - 1)!a^{p-1} \equiv (p - 1)! \pmod{p} \]
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- Canceling out \( (p - 1)! \) from both sides, we have the theorem!
Primality Testing

Given a number \( N \), we can use Fermat’s theorem as a probabilistic test to see if it is prime:

- if \( a^{N-1} \not\equiv 1 \pmod{N} \) then \( N \) is not prime
- Repeat with different values of \( a \) to gain more confidence
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Given a number $N$, we can use Fermat’s theorem as a probabilistic test to see if it is prime:

- if $a^{N-1} \not\equiv 1 \pmod{N}$ then $N$ is not prime
- Repeat with different values of $a$ to gain more confidence

**Question:** If $N$ is *not* prime, what is the probability that the above procedure will fail?
Primality Testing

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**Question:** If $N$ is *not* prime, what is the probability that the above procedure will fail?

- For Carmichael’s numbers, the probability is 1 — but ignore this for now, since these numbers are very rare.
- For other numbers, we can show that the above procedure works with probability 0.5
Lemma

If \( a^{N-1} \not\equiv 1 \pmod{N} \) for a relatively prime to \( N \), then it holds for at least half the choices of \( a < N \).
**Lemma**

If $a^{N-1} \not\equiv 1 \pmod{N}$ for a relatively prime to $N$, then it holds for at least half the choices of $a < N$.

- If there is no $b$ such that $b^{N-1} \equiv 1 \pmod{N}$, then we have nothing to prove.
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- Note \( c^{N-1} \equiv a^{N-1}b^{N-1} \equiv a^{N-1} \neq 1 \)
Lemma

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- Note \( c^{N-1} \equiv a^{N-1}b^{N-1} \equiv a^{N-1} \not\equiv 1 \)
- Thus, for every \( b \) for which Fermat’s test is satisfied, there exists a \( c \) that does not satisfy it.
  - Moreover, since \( a \) is relatively prime to \( N \), \( ab \not\equiv ab' \) unless \( b \equiv b' \).
Primality Testing

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- Thus, for every \( b \) for which Fermat’s test is satisfied, there exists a \( c \) that does not satisfy it.
  - Moreover, since \( a \) is relatively prime to \( N \), \( ab \not\equiv ab' \) unless \( b \equiv b' \).
- Thus, at least half of the numbers \( x < N \) that are relatively prime to \( N \) will fail Fermat’s test.
Primality Testing

Figure 1.7 An algorithm for testing primality.

function primality (N)
Input: Positive integer N
Output: yes/no
Pick a positive integer a < N at random
if \( a^{N-1} \equiv 1 \pmod{N} \):
    return yes
else:
    return no

Is \( a^{N-1} \equiv 1 \pmod{N} \)?

Fermat’s test

- When Fermat’s test returns “prime” \( Pr[N \text{ is not prime}] < 0.5 \)
- If Fermat’s test is repeated for \( k \) choices of \( a \), and returns “prime” in each case, \( Pr[N \text{ is not prime}] < 0.5^k \)
- In fact, 0.5 is an upper bound. Empirically, the probability has been much smaller.
Empirically, on numbers less than 25 billion, the probability of Fermat’s test failing to detect non-primes (with $a = 2$) is more like 0.00002.

This probability decreases even more for larger numbers.
Prime number generation

Lagrange’s Prime Number Theorem
For large $N$, primes occur approx. once every $\log N$ numbers.
Lagrange’s Prime Number Theorem

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Generating Primes

- Generate a random number
- Probabilistically test it is prime, and if so output it
- Otherwise, repeat the whole process
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What is the complexity of this procedure?

$O((\log_2 N) \log N)$ multiplications on $\log N$ bit numbers

If $N$ is not prime, should we try $N+1$, $N+2$, etc. instead of generating a new random number?
No, it is not easy to decide when to give up.
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Rabin-Miller Test

- Works on Carmichael’s numbers
- For prime number test, we consider only odd $N$, so $N - 1 = 2^t u$ for some odd $u$
- Compute

$$a^u, a^{2u}, a^{4u}, \ldots, a^{2^t u} = a^{N-1}$$

- If $a^{N-1}$ is not 1 then we know $N$ is composite.
- Otherwise, we do a follow-up test on $a^u, a^{2u}$ etc.
  - Let $a^{2^r u}$ be the first term that is equivalent to 1.
  - If $r > 0$ and $a^{2^{r-1} u} \not\equiv -1$ then $N$ is composite

- This combined test detects non-primes with a probability of at least 0.75 for all numbers.
Global Min-cut in Undirected Graphs

- Compute the minimum number of edges that need to be severed to disconnect a graph
- Yields the edge-connectivity of the graph

A multigraph whose minimum cut has three edges.
Deterministic Global Min-cut

- Replace each undirected edge by two (opposing) directed edges
- Pick a vertex \( s \)
- for each \( t \) in \( V \) compute the minimum \( s - t \) cut
- The smallest among these is the global min-cut
- Repeating min-cut \( O(|V|) \) times, so it is expensive and complex.
Randomized global min-cut

- Relies on repeated “collapsing” of edges, illustrated below
  - Pick a random edge \((u, v)\), and delete it
  - Replace \(u\) and \(v\) by a single vertex \(uv\)
  - Replace each edge \((x, u)\) by \((x, uv)\)
  - Replace each edge \((x, v)\) by \((x, uv)\)

- Note: edges maintain their identity during this process
Randomized global min-cut

```plaintext
GuessMinCut(V, E)

if |V| = 2 then
    return the only cut remaining

Pick an edge at random and collapse it to get V', E'

return GuessMinCut(V', E')
```

Does this algorithm make sense? Why should it work?

Basic idea: Only a small fraction of edges belong to the min-cut, reducing the likelihood of them being collapsed. Still, when almost every edge is being collapsed, how likely is it that min-cut edges will remain?
Randomized global min-cut

\[ \text{GuessMinCut}(V, E) \]

\[
\text{if } |V| = 2 \text{ then}
\]

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\text{return the only cut remaining}
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**GuessMinCut Correctness Probability**

- If min-cut has $k$ edges, then every node has min degree $k$
- So, there are $nk/2$ edges
**GuessMinCut Correctness Probability**

- If min-cut has $k$ edges, then every node has min degree $k$
- So, there are $nk/2$ edges
- The likelihood of collapsing them in the first step is $2/n$
  - The likelihood of preserving min-cut edges is $(n - 2)/n$
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- We thus have the following recurrence for likelihood of preserving min-cut edges in the final solution:

$$
P(n) \geq \frac{n-2}{n} \cdot P(n-1) \geq \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdots \frac{2}{4} \cdot \frac{1}{3} = \frac{2}{n(n-1)}
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So, the probability of being wrong is high

- by repeating it \( O(n^2 \ln n) \) times, we reduce it to \( 1/n^c \).
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So, the probability of being wrong is high
- by repeating it \( O(n^2 \ln n) \) times, we reduce it to \( 1/n^c \).

Overall runtime is \( O(n^4 \ln n) \), which is hardly impressive.
If a single random choice yields unsatisfactory results, try making two choices and pick the better of two.
Power of Two Random Choices

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*Example applications*

**Balls and bins:** Maximum occupancy comes down from $O(\log n / \log \log n)$ to $O(\log \log n)$
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**Quicksort:** Significantly increase odds of a balanced split if you pick three random elements and use their median as pivot
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**Balls and bins:** Maximum occupancy comes down from $O(\log n / \log \log n)$ to $O(\log \log n)$

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**Load balancing:** Random choice does not work well if different tasks take different time. Making two choices and picking the lighter loaded of the two can lead to much better outcomes
Power of Two Random Choices for Min-cut

- Divide random collapses into two phases
  - An initial “safe” phase that shrinks the graph to $1 + \frac{n}{\sqrt{2}}$ nodes
    - Probability of preserving min-cut is
      \[
      \frac{(\frac{n}{\sqrt{2}})(\frac{n}{\sqrt{2}} + 1)}{\frac{n}{2}(n - 1)} \geq \frac{1}{2}
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    \frac{(\frac{n}{\sqrt{2}})(\frac{n}{\sqrt{2}} + 1)}{\frac{n}{\sqrt{2}}(n - 1)} \geq \frac{1}{2}
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- A second “unsafe” phase that is run twice, and the smaller min-cut is picked
Power of Two Random Choices for Min-cut

- A single run of unsafe phase is simply a recursive call
Power of Two Random Choices for Min-cut

- A single run of unsafe phase is simply a recursive call
- A kind-of-divide and conquer with power-of-two
  - Since input size decreases with each level of recursion, total time is reduced in spite of exponential increase in number of iterations

We get the following recurrence for correctness probability:

\[ P(n) \geq 1 - \frac{1}{2} P \left( \sqrt{2} n + 1 \right) \]

which yields a result of \( \Omega\left(\frac{1}{\log n}\right) \)

Need \( O\left(\log(2\sqrt{n})\right) \) repetitions to obtain low error rate

For runtime, we have the recurrence

\[ T(n) = O(n^2) + 2T(\sqrt{2}n + 1) = O(n^2 \log n) \]

Incl. \( \log(2\sqrt{n}) \) iterations, total runtime is \( O(n^2 \log(3\sqrt{n})) \)!
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    - Since input size decreases with each level of recursion, total time is reduced in spite of exponential increase in number of iterations

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  \[
P(n) \geq 1 - \left( 1 - \frac{1}{2} P \left( \frac{n}{\sqrt{2}} + 1 \right) \right)^2
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  - Incl. \( \log^2 n \) iterations, total runtime is \( O(n^2 \log^3 n)! \)