CSE 548: Algorithms

Dynamic Programming

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Overview

- Another approach for *optimization problems*, more general and versatile than greedy algorithms.

- **Optimal substructure** The optimal solution contains optimal solutions to subproblems.

- **Overlapping subproblems.** Typically, the same subproblems are solved repeatedly.

- Solve subproblems in *a certain order*, and *remember solutions* for later reuse.
Topics

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   Overview

2. LIS
   DAG Formulation
   Algorithm

3. Knapsack
   Knapsack w/ Repetition
   0-1 Knapsack

4. Chain MM

5. LCS
   Defn
   Towards Soln.
   Variations
   Seq. Alignment
   UNIX apps
DAGs and Dynamic Programming

- *Canonical way to represent dynamic programming*
  
  **Nodes** in the DAG represent subproblems
  
  **Edges** capture dependencies between subproblems
  
  **Topological sorting** solves subproblems in the right order
  
  **Remember** subproblem solutions to avoid recomputation
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- For problems in other domains, DAGs are implicit, as is the topological sort.
Longest Increasing Subsequence

**Definition**

Given a sequence $a_1, a_2, \ldots, a_n$, its LIS is a sequence

$$a_{i_1}, a_{i_2}, \ldots, a_{i_k}$$

that maximizes $k$ subject to $i_j < i_{j+1}$ and $a_i \leq a_{i+1}$.

---

5  2  8  6  3  6  9  7

---

In this example, the arrows denote transitions between consecutive elements of the optimal solution. More generally, to better understand the solution space, let's create a graph of all permissible transitions: establish a node $i$ for each element $a_i$, and add directed edges $(i, j)$ whenever it is possible for $a_i$ and $a_j$ to be consecutive elements in an increasing subsequence, that is, whenever $i < j$ and $a_i < a_j$ (Figure 6.2).
Casting LIS problem using a DAG

**Nodes:** represent elements in the sequence

**Edges:** connect an element to all followers that are larger

**Topological sorting:** sequence already topologically sorted

**Remember:** Using an array \( L[1..n] \)

---

**Figure 6.2** The dag of increasing subsequences.

5 2 8 3 9 7 6 6

Notice that (1) this graph \( G = (V, E) \) is a dag, since all edges \((i, j)\) have \( i < j \), and (2) there is a one-to-one correspondence between increasing subsequences and paths in this dag. Therefore, our goal is simply to find the longest path in the dag!

Here is the algorithm:

```plaintext
for j = 1, 2, ..., n:
    L[j] = 1 + \max\{L[i] : (i, j) \in E\}

return \max_j L[j]
```

\( L[j] \) is the length of the longest path—the longest increasing subsequence—ending at \( j \) (plus 1, since strictly speaking we need to count nodes on the path, not edges). By reasoning in the same way as we did for shortest paths, we see that any path to node \( j \) must pass through one of its predecessors, and therefore \( L[j] \) is 1 plus the maximum \( L[i] \) value of these predecessors.

If there are no edges into \( j \), we take the maximum over the empty set, zero. And the final answer is the largest \( L[j] \), since any ending position is allowed.

This is dynamic programming. In order to solve our original problem, we have defined a collection of subproblems \( \{L[j] : 1 \leq j \leq n\} \) with the following key property that allows them to be solved in a single pass:

\[ (*) \]

There is an ordering on the subproblems, and a relation that shows how to solve a subproblem given the answers to “smaller” subproblems, that is, subproblems that appear earlier in the ordering.

In our case, each subproblem is solved using the relation

\[ L[j] = 1 + \max\{L[i] : (i, j) \in E\} \]

an expression which involves only smaller subproblems. How long does this step take? It requires the predecessors of \( j \) to be known; for this the adjacency list of the reverse graph \( G^R \), constructible in linear time (recall Exercise 3.5), is handy. The computation of \( L[j] \) then takes time proportional to the indegree of \( j \), giving an overall running time linear in \(|E|\). This is at most \( O(n^2) \), the maximum being when the input array is sorted in increasing order. Thus the dynamic programming solution is both simple and efficient.

There is one last issue to be cleared up: the \( L \)-values only tell us the length of the optimal subsequence, so how do we recover the subsequence itself? This is easily managed with the...
Algorithm for LIS

\[ \text{LIS}(E) \]

\[
\text{for } j = 1 \text{ to } n \text{ do}
\]

\[
L[j] = 1 + \max_{(i,j) \in E} L[i]
\]

\[
\text{return } \max_{j=1}^{n} L[j]
\]

Notice that (1) this graph \( G = (V, E) \) is a dag, since all edges \((i, j)\) have \( i < j \), and (2) there is a one-to-one correspondence between increasing subsequences and paths in this dag. Therefore, our goal is simply to find the longest path in the dag!

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\[ \text{Figure 6.2} \text{The dag of increasing subsequences.} \]

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Correctness: Straight-forward

Complexity: What is it? Can it be improved?
Algorithm for LIS

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Knapsack Problem (Recap)

- You have a choice of items you can pack in the sack
- Maximize value of sack, subject to a weight limit of $W$

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<td>5</td>
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**Fractional knapsack:** Fractional quantities acceptable

**0-1 knapsack:** Take all of one item or none at all

**Knapsack w/ repetition:** Take any integral number of items.
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No polynomial solution for the last two, but dynamic programming can solve them in *pseudo-polynomial* time of $O(nW)$. 
Knapsack w/ repetition: Identify subproblems

- Consider subproblems by reducing the weight
- Compute $K(W)$ in terms of $K(W')$ for $W' < W$
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- **Optimal substructure:** If $K(W)$ is the optimal solution and it includes item $i$, then $K(W) = K(W - w_i) + v_i$
Knapsack w/ repetition

\[ \text{KnapWithRep}(w, v, n, W) \]

\[
\begin{align*}
K[0] &= 0 \\
\text{for } w = 1 \text{ to } W \text{ do} \\
K[w] &= \max_{1 \leq i \leq n, w[i] \leq w} (K[w - w[i]] + v[i]) \\
\text{return } K[W]
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  - If you construct the dag explicitly, you will see that we are looking for the longest path!

Runtime:
- Outer loop iterates \( W \) times,
- \( \max \) takes \( O(n) \) time, for a total of \( O(nW) \) time
- Not polynomial: input size logarithmic (not linear) in \( W \).
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Previous algorithm does not work. We need to keep track of which items have been used up.
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Key idea: Define 2-d array $K[u,j]$ which computes optimal value for weight $u$ achievable using items 1..$j$
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So, fill up the array $K$ as $j$ goes from 1 to $n$

For each $j$, fill $K$ as $u$ goes from 1 to $W$
0-1 Knapsack Algorithm

\( \text{Knap01}(w, v, n, W) \)

\[
K[u, 0] = K[0, j] = 0, \forall 1 \leq u \leq W, 1 \leq j \leq n
\]

for \( j = 1 \) to \( n \) do

    for \( u = 1 \) to \( W \) do

        if \( w[j] > u \) then \( K[u, j] = K[u, j-1] \)

        else \( K[u, j] = \max(K[u, j-1], K[u-w[j], j-1] + v[j]) \)

return \( K[W, n] \)
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\quad K[u-w[j], j-1] + v[j])

\textbf{return } K[W, n]

\textbf{Runtime: } As compared to unbounded knapsack, we have a nested loop here, but the inner loop now executes in } O(1) \text{ time. So runtime is still } O(nW)
Recursive formulation of Dynamic programming

- Recursive formulation can often simplify algorithm presentation, avoiding need for explicit scheduling
  - Dependencies between subproblems can be left implicit: an equation such as
    \[ K[w] = K[w - w[j]] + v[j] \]
  - A call to compute \( K[w] \) will automatically result in a call to compute \( K[w - w[j]] \) because of dependency
  - *Can avoid solving (some) unneeded subproblems*
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- *Can avoid solving (some) unneeded subproblems*

*Memoization*: Remember solutions to function calls so that repeat invocations can use previously returned solutions.
Recursive 0-1 Knapsack Algorithm

\[ \text{BestVal01}(u, j) \]

- if \( u = 0 \) or \( j = 0 \) return 0
- if \( w[j] > u \) return \( \text{BestVal01}(u, j-1) \)
- else return \( \max(\text{BestVal01}(u, j-1), v[j] + \text{BestVal01}(u-w[j], j-1)) \)

- Much simpler in structure than iterative version
- Unneeded entries are not computed, e.g. \( \text{BestVal01}(3, _) \) when all weights involved are even
- **Exercise:** Write a recursive version of ChainMM.

**Note:** \( m_i \)'s give us the dimension of matrices, specifically, \( M_i \) is an \( m_{i-1} \times m_i \) matrix
Recursive 0-1 Knapsack Algorithm

**BestVal01**(*u*, *j*)

- **if** *u* = 0 **or** *j* = 0 **return** 0
- **if** *w[j]* > *u* **return** **BestVal01**(*u*, *j* − 1)
- **else return** **max**(**BestVal01**(*u*, *j* − 1), *v[j]* + **BestVal01**(*u* − *w[j]*, *j* − 1))

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**Exercise:** Write a recursive version of ChainMM.

**Note:** *m*_i’s give us the dimension of matrices, specifically, *M*_i is an *m*_i−1 × *m*_i matrix

**Complexity:** **O**(*n*^3^)
Key step in Dyn. Prog.: Identifying subproblems

i. The input is $x_1, x_2, \ldots, x_n$ and a subproblem is $x_1, x_2, \ldots, x_i$.

\begin{align*}
\begin{array}{ccccccc}
  x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \\
\end{array}
\end{align*}

The number of subproblems is therefore linear.

ii. The input is $x_1, \ldots, x_n$, and $y_1, \ldots, y_m$. A subproblem is $x_1, \ldots, x_i$ and $y_1, \ldots, y_j$.

\begin{align*}
\begin{array}{ccccccc}
  x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \\
  y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\
\end{array}
\end{align*}

The number of subproblems is $O(mn)$.

iii. The input is $x_1, \ldots, x_n$ and a subproblem is $x_i, x_{i+1}, \ldots, x_j$.

\begin{align*}
\begin{array}{ccccccc}
  x_1 & x_2 & | & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \\
\end{array}
\end{align*}

The number of subproblems is $O(n^2)$.

iv. The input is a rooted tree. A subproblem is a rooted subtree.
Chain Matrix Multiplication

(a) $A \times (B \times C) \times D$

(b) $A (B \times C) \times D$

(c) $A \times (B \times C) \times D$

(d) $(A \times (B \times C)) \times D$

<table>
<thead>
<tr>
<th>Parenthesization</th>
<th>Cost computation</th>
<th>Cost</th>
</tr>
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<tbody>
<tr>
<td>$A \times ((B \times C) \times D)$</td>
<td>$20 \cdot 1 \cdot 10 + 20 \cdot 10 \cdot 100 + 50 \cdot 20 \cdot 100$</td>
<td>120, 200</td>
</tr>
<tr>
<td>Greedy</td>
<td>$(A \times (B \times C)) \times D$</td>
<td>$20 \cdot 1 \cdot 10 + 50 \cdot 20 \cdot 10 + 50 \cdot 10 \cdot 100$</td>
</tr>
<tr>
<td></td>
<td>$(A \times B) \times (C \times D)$</td>
<td>$50 \cdot 20 \cdot 1 + 1 \cdot 10 \cdot 100 + 50 \cdot 1 \cdot 100$</td>
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Chain MM: Formulating Optimal Solution

Consider outermost multiplication: 

$$(M_1 \times \cdots \times M_j) \times (M_{j+1} \times \cdots \times M_n)$$  — we could compute $j$ using dynamic programming

Optimal substructure: Note that the optimal solution for 

$$(M_1 \times \cdots \times M_j) \times (M_{j+1} \times \cdots \times M_n)$$  must rely on optimal solutions to $M_1 \times \cdots \times M_j$ and $M_{j+1} \times \cdots \times M_n$  — or else we could improve the overall solution still

Cost function: This suggests a cost function $C[k, l]$ to denote the optimal cost of $M_k \times \cdots \times M_l$
Chain MM: Formulating Optimal Solution

Subproblems correspond to one of the subtrees
Chain MM: Formulating Optimal Solution

Subproblems correspond to one of the subtrees

Since order of multiplications can’t be changed, each subtree must correspond to a “substring” of multiplications, i.e., $M_k \times \cdots \times M_l$
Chain MM Algorithm

\[ \text{chainMM}(m, n) \]

\[
C[i, i] = 0 \forall 1 \leq i \leq n
\]

\[ \text{for } s = 1 \text{ to } n - 1 \text{ do} \]

\[ \text{for } k = 1 \text{ to } n - s \text{ do} \]

\[ l = k + s \]

\[
C[k, l] = \min_{k \leq i < l}(C[k, i] + C[i + 1, l] + m_{k-1} * m_i * m_l)
\]

\[ \text{return } C[1, n] \]

- Recall: subproblems correspond to substrings: \( M_k \times \cdots \times M_l \)
- We iterate in increasing order of substring length
  - \( s \) goes from 1 to \( n - 1 \) and represents substring length minus 1.
- Substrings of same lengths are considered left to right,
  - \( k \) goes from 1 to \( n - s \) and represents the starting position of substring
Chain MM Algorithm

\( \text{chainMM}(m, n) \)

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\begin{align*}
C[i, i] &= 0 \ \forall 1 \leq i \leq n \\
\text{for } s = 1 \text{ to } n - 1 \text{ do} \\
\quad \text{for } k = 1 \text{ to } n - s \text{ do} \\
\quad \quad l &= k + s \\
\quad C[k, l] &= \min_{k \leq i < l} (C[k, i] + C[i + 1, l] + m_{k-1} \times m_i \times m_l) \\
\text{return } C[1, n]
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Chain MM Algorithm

`chainMM(m, n)`

- `C[i, i] = 0 ∀1 ≤ i ≤ n`
- `for s = 1 to n - 1 do`
  - `for k = 1 to n - s do`
    - `l = k + s`
    - `C[k, l] = min_{k ≤ i < l}(C[k, i] + C[i + 1, l] + m_{k-1} * m_i * m_l)`
- `return C[1, n]`

Note: `m_i`'s give us the dimension of matrices, specifically, `M_i` is an `m_{i-1} × m_i` matrix

Complexity: `O(n^3)`
Subsequence

**Definition**

A sequence $a[1..m]$ is a subsequence of $b[1..n]$ occurring at position $r$ if there exist $i_1, ..., i_k$ such that $a[r..(r+l-1)] = b[i_1]b[i_2] \cdots b[i_l]$, where $i_j < i_{j+1}$.
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The relative order of elements is preserved in a subsequence, but unlike a substring, the elements need not be contiguous.

*Example:* $BDEFHJ$ is a subsequence of $ABCDEFGHIJK$
Longest Common Subsequence

Definition (LCS)

The LCS of two sequences $x[1..m]$ and $y[1..n]$ is the longest sequence $z[1..k]$ that is a subsequence of both $x$ and $y$. 
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By aligning elements of $z$ with the corresponding elements of $x$ and $y$, we can compare $x$ and $y$.
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\[
\begin{align*}
x & : \ P \ R \ O \ F \ - \ E \ S \ S \ O \ R \\
z & : \ P \ R \ O \ F \ - \ E \ S \ - \ - \ R \\
y & : \ P \ R \ O \ F \ F_{ins} \ E \ S \ -_{del} \ U_{sub} \ R
\end{align*}
\]

to identify *edit* operations (insert/delete/substitute) operations needed to map $x$ to $y$
Edit (Levenshtein) distance

**Definition (ED)**

Given sequences $x$ and $y$ and functions $I$, $D$ and $S$ that associate costs with each insert, delete and substitute operations, what is the minimum cost of any edit sequence that transforms $x$ into $y$. 

Applications

- Spell correction (Levenshtein automata)
- `diff` in the context of version control, reconcile/merge concurrent updates by different users.
- DNA sequence alignment, evolutionary trees and other applications in computational biology.
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Towards a dynamic programming solution (1)

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**Exponential Polynomial**

The subproblem above can be represented as $E[7, 5]$.

$E[i, j]$ represents the edit distance of $x[1..i]$ and $y[1..j]$.
Towards a dynamic programming solution (2)

For $E[k, l]$, consider the following possibilities:
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    - $E[k, l] = E[k - 1, l] + DC(x[k])$
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    $$E[k, l] = E[k, l - 1] + IC(y[l])$$
  - extend $E[k - 1, l - 1]$ by substituting $x[k]$ with $y[l]$:
    $$E[k, l] = E[k - 1, l - 1] + SC(x[k], y[l])$$
Towards a dynamic programming solution (3)

\[ E[k, l] = \min(E[k-1, l] + DC(x[k]), \quad // \downarrow \]
\[ E[k, l-1] + IC(y[l]), \quad // \rightarrow \]
\[ E[k-1, l-1] + SC(x[k], y[l])) \quad // \downarrow \]

\[ E[0, l] = \sum_{i=1}^{l} IC(y[i]) \]
\[ E[k, 0] = \sum_{i=1}^{k} DC(x[i]) \]

Edit distance = \( E[m, n] \)

(Recall: \( m \) and \( n \) are lengths of strings \( x \) and \( y \))
Towards a dynamic programming solution (4)

![Diagram of a grid with subproblems and arrows indicating precedence constraints.](image)

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E[k - 1, l - 1] + SC(x[k], y[l])) \quad \text{// } \downarrow
\]
Towards a dynamic programming solution (5)

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Variations

Approximate prefix:

Is $y$ approx. prefix of $x$?
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Initialize $E[k, 0] = 0$, use $E[m, n]$ to determine if $y$ is an approximate suffix of $x$
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More variations

Supporting transpositions:

\[
E[k-2, l-2] + TC(x[k-1], y[l-1])
\]

where \( TC \) is a small value for transposed characters, and \( \infty \) otherwise.
Supporting transpositions:

Use a fourth term within *min*:

\[ E[k - 2, l - 2] + TC(x[k - 1]x[k], y[l - 1]y[l]) \]

where *TC* is a small value for transposed characters, and \( \infty \) otherwise.
Similarity Vs Edit-distance

**Edit-distance** cannot be interpreted on its own, and needs to take into account the lengths of strings involved.

**Similarity** can stand on its own.

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\[
S[0, l] = \\
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\bullet \text{SC}(r, r) \text{ should be negative, while IC and DC should be positive.} \\
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Global alignment (DNA, proteins, ...)

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- Quadratic time still too slow for sequence alignment.
Local alignment

- Aimed at identifying local regions of similarity, specifically, the best matches between subsequences of $x$ and $y$
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A new term is introduced within $\max$, namely, zero. This means that costs can never become negative. In other words, a subsequence does not incur costs because of mismatches preceding the subsequence. This change enables regions of similarity to stand out as positive scores.

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Improvements to ED Algorithm

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**Better overall performance:** \(O(md)\) space and runtime if the max. distance \(\leq d\).

In the interest of time, we won’t cover these extensions. They are fairly involved, but not necessarily hard.
LCS application: UNIX diff

Each line is considered a “character:”

- Number of lines far smaller than number of characters
- Difference at the level of lines is easy to convey to users
- Much higher degree of confidence when things line up. Leads to better results on programs.

But does not work that well on document types where line breaks are not meaningful, e.g., text files where each paragraph is a line.

Aligns lines that are preserved.

- The edits are then printed in the familiar “diff” format.
Software patches often distributed as “difs.” Programs such as `patch` can apply these patches to source code or any other file.
LCS applications: version control, patch,…

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Concurrent updates in version control systems are resolved using LCS.
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Concurrent updates in version control systems are resolved using LCS.

- Let $x$ be the version in the repository
- Suppose that user $A$ checks it out, edits it to get version $y$
- Meanwhile, $B$ also checks out $x$, edits it to $z$.
- If $x \rightarrow y$ edits target a disjoint set of locations from those targeted by the $x \rightarrow z$ edits, both edits can be committed; otherwise a conflict is reported.
A general approach for *optimization problems*

*Applicable in the presence of:*

- Optimal substructure
- A natural ordering among subproblems
- Numerous subproblems (often, exponential), but only some (polynomial number) are distinct