CSE 548: Algorithms

Amortized Analysis

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Amortized Analysis

Amortization

The spreading out of capital expenses for intangible assets over a specific period of time (usually over the asset’s useful life) for accounting and tax purposes.

- A clever trick used by accountants to average large one-time costs over time.
- In algorithms, we use amortization to spread out the cost of expensive operations.

- Example: Re-sizing a hash table.
Topics

1. Intro
   Motivation
2. Aggregate
3. Charging
4. Potential
5. Table resizing
6. Disjoint sets
   Amortized Rehashing
   Vector and String Resizing
   Inverted Trees
   Union by Depth
   Threaded Trees
   Path compression
Some operations have high worst-case cost, but we can show that the worst case does not occur every time.

In this case, we can average the costs to obtain a better bound.
Summation or Aggregate Method

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**Summation**

Let $T(n)$ be the worst-case running time for executing a sequence of $n$ operations. Then the amortized time for each operation is $T(n)/n$. 
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In this case, we can average the costs to obtain a better bound.

**Summation**

Let $T(n)$ be the worst-case running time for executing a sequence of $n$ operations. Then the amortized time for each operation is $T(n)/n$.

*Note:* We are not making an “average case” argument about inputs. *We are still talking about worst-case performance.*
What is the worst-case runtime of \textit{incr}?

\begin{verbatim}
Incr(B[0..])
i = 0
while B[i] = 1
B[i] = 0
i++
B[i] = 1
\end{verbatim}
Summation Example: Binary Counter

- What is the worst-case runtime of \textit{incr}?
  - Simple answer: $O(\log n)$, where $n =$ \# of \textit{incr}'s performed

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Incr(B[0..])

\begin{align*}
i &= 0 \\
\text{while } B[i] &= 1 \\
B[i] &= 0 \\
i &= i + 1 \\
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```
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What is the amortized runtime for $n$ `incr`’s?

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  - Easy to see that an `incr` will touch $B[i]$ once every $2^i$ operations.
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  - Easy to see that an \textit{incr} will touch \(B[i]\) once every \(2^i\) operations.
  - Number of operations is thus

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\sum_{i=0}^{\log n} \frac{1}{2^i} = 2n
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$$n \sum_{i=0}^{\log n} \frac{1}{2^i} = 2n$$

- Thus, amortized cost per `incr` is $O(1)$
Charging Method

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  - *Important: ensure you have charged enough.*
    - We have satisfied this: a bit can be flipped from 1 to 0 only once after it is flipped from 0 to 1.
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- Pays for the cost of later flipping the 1-bit to 0-bit.
  - *Important: ensure you have charged enough.*
    - We have satisfied this: a bit can be flipped from 1 to 0 only once after it is flipped from 0 to 1.
- Now we ignore costs of 1 to 0 flips in the algorithm
  - There is only one 0-to-1 bit flipping per call of `incr`!
  - So, `incr` only costs 2 units for each invocation!
Stack Example

Consider a stack with two operations:

- **push(x)**: Push a value $x$ on the stack
- **pop(k)**: Pop off the top $k$ elements
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  - push(x): Push a value x on the stack
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- What is the cost of a mix of n push and pop operations?
- **Key problem**: Worst-case cost of a pop is $O(n)$!
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  - `push(x)`: Push a value $x$ on the stack
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- What is the cost of a mix of $n$ push and pop operations?

- *Key problem*: Worst-case cost of a `pop` is $O(n)$!

- *Solution*:
  - Charge 2 units for each push: covers the cost of pushing, and also the cost of a subsequent pop
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- \textit{Key problem}: Worst-case cost of a \texttt{pop} is \( O(n) \)!

- \textit{Solution}:
  - Charge 2 units for each push: covers the cost of pushing, and also the cost of a subsequent pop
  - A pushed item can be popped only once, so we have charged enough
  - Now, ignore \texttt{pop}'s altogether, and trivially arrive at \( O(1) \) amortized cost for the sequence of push/pop operations!
Potential Method

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  - as the data structure changes and “releases” stored energy
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  - as the data structure changes and “releases” stored energy

- A more sophisticated technique that allows “charges” or “taxes” to be stored within nodes in a data structure and used subsequently at a later time.
Potential Method: Illustration

Stack:

- Each push costs 2 units because a push increases potential energy by 1.
- Pops can use the energy released by reduction in stack size!
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- Each push costs 2 units because a push increases potential energy by 1.
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Counter:
- Define potential as the number one 1-bits
- Changing a 0 to 1 costs 2 units, one for the operation and one to pay for increase in potential
- Changes of 1 to 0 can now be paid by released potential.
Hash Tables

- To provide expected constant time access, collisions need to be limited
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2. Quit complaining, bite the bullet, and rehash as needed;
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  - But this requires all entries to be deleted from current table and inserted into a table that is larger — *a very expensive operation*.

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1. Try to guess the table size right; if you guessed wrong, put up with the pain of low performance.
2. Quit complaining, bite the bullet, and rehash as needed;
3. *Amortize*: Rehash as needed, *and* prove that it does not cost much!
Amortized Rehashing

Amortize the cost of rehashing over other hash table operations
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**Approach 1:** Rehash after a large number (say, 1K) operations.

Total cost of 1K ops = 1K for the ops + 1K for rehash = 2K
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So, amortized cost is just 2!
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Are we done?
Amortized Rehash (2)

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Consider total cost after 2K, 3K, and 4K operations:

\[ T(2K) = 2K + 1K \text{ (first rehash)} + 2K \text{ (second rehash)} = 5K \]
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Hmmm. This is growing like \( n^2 \), so amortized cost will be \( O(n) \)
Need to try a different approach.
Amortized Rehash (3)

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You invoke $N$ insert operations, then rehash to a $2N$ table.

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The general recurrence is $T(n) = T(n/2) + 1.5n$, which is linear.

So, amortized cost is constant!
Amortized Rehash (4)

Alternatively, we can think in terms of charging.
Each insert operation can be charged 3 units of cost:
- One for the insert operation
- One for rehashing of this element at the end of this run of inserts
- One for rehashing an element that was already in the hash table when this run began
Amortized Rehash (4)

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A run contains as many elements as the hash table at the beginning of run — so we have accounted for all costs. Thus, rehashing

- increases the costs of insertions by a factor of 3.
- lookup costs are unchanged.
Amortized Rehash (5)

- Alternatively, we can think in terms of *potential*.
- Hash table as a spring: as more elements are inserted, the spring has to be compressed to make room.
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- Hash table as a spring: as more elements are inserted, the spring has to be compressed to make room.
- Let $|H|$ denote the capacity and $\alpha$ the occupancy of $H$
- Define potential as 0 when $\alpha \leq 0.5$ and $2(\alpha - 0.5)|H|$ otherwise.
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- Immediately after resize, let the hash table capacity be $k$. Note $\alpha \leq 0.5$ so potential is $0$. 
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- Each insert (after $\alpha$ reaches 0.5) costs 3 units: one for the operation, and 2 for the increase in potential.
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- Each insert (after $\alpha$ reaches 0.5) costs 3 units: one for the operation, and 2 for the increase in potential.

- When $\alpha$ reaches 1, the potential is $2k$. After resizing to $2k$, potential falls to 0, and the released $2k$ cost pays for rehashing $2k$ elements.
What if we increase the size by a factor less than 2?

- Is there a threshold \( t > 1 \) such that expansion by a factor less than \( t \) won’t yield amortized constant time?

What happens if we want to support both deletes and inserts, and want to make sure that the table never uses more than \( k \) times the actual number of elements?

- Is there a minimum value of \( k \) for which this can be achieved?

- Do you need a different threshold for expansion and contraction? Are there any constraints on the relationship between these two thresholds to ensure amortized constant time?
Amortized performance of Vectors vs Lists

Linked lists: Data structures of choice if you don’t know the # of elements in advance.

- Space inefficient: 2x or more memory for very small objects.
- Poor cache performance: Pointer chasing is cache unfriendly.
- Sequential access: No fast access to \( k \)th element.

Vectors: Dynamically-sized arrays have none of these problems. But resizing is expensive.
- Is it possible to achieve good amortized performance?
- When should the vector be expanded/contracted?
- What operations can we support in constant amortized time? Inserts? insert at end? concatenation?

Strings: We can raise similar questions as Vectors.
Disjoint Sets

- Represent disjoint sets as “inverted trees”
- Each element has a parent pointer $\pi$
- To compute the union of set $A$ with $B$, simply make $B$’s root the parent of $A$’s root.

A directed-tree representation of two sets $\{B, E\}$ and $\{A, C, D, F, G, H\}$.
Disjoint Sets (2)

procedure makeset(x)
\[ \pi(x) = x \]
\[ \text{rank}(x) = 0 \]

function find(x)
while \( x \neq \pi(x) \):
\[ x = \pi(x) \]
return \( x \)
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procedure union(x, y)
\[ r_x = \text{find}(x) \]
\[ r_y = \text{find}(y) \]
\[ \pi(r_y) = r_x \]
Disjoint Sets (2)

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Complexity
- makeset takes \( O(1) \) time
Disjoint Sets (2)

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- union takes \( O(1) \) time on a root element; in the worst case, its complexity matches \text{find}. 
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function find(x)
while x \neq \pi(x): \ x = \pi(x)
return x

procedure union(x, y)
rx = find(x)
y = find(y)
\pi(ry) = rx
```

Complexity
- makeset takes $O(1)$ time
- find takes time equal to depth of set: $O(n)$ in the worst case.
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Amortized complexity
- Can you construct a worst-case example, where $N$ operations take $O(N^2)$ time?
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\[x = \pi(x)\]
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procedure union(x, y)
\(r_x = \text{find}(x)\)
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\[\pi(r_y) = r_x\]

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Amortized complexity
- Can you construct a worst-case example, where \(N\) operations take \(O(N^2)\) time?
- Can we improve this?
Disjoint Sets with Union by Depth

```
procedure makeset(x)
  π(x) = x
  rank(x) = 0

function find(x)
  while x ≠ π(x)
    x = π(x)
  return x
```

```
procedure union(x, y)
  r_x = find(x)
  r_y = find(y)
  if r_x = r_y: return
  if rank(r_x) > rank(r_y):
    π(r_y) = r_x
    π(r_y) = r_y
  else:
    π(r_x) = r_y
    rank(r_y) = rank(r_y) + 1
```
Disjoint Sets with Union by Depth (2)

**Figure 5.6** A sequence of disjoint-set operations. Superscripts denote rank.

After makeset(A), makeset(B), ..., makeset(G):

\[
\begin{array}{cccccccc}
A^0 & B^0 & C^0 & D^0 & E^0 & F^0 & G^0 \\
\end{array}
\]

After union(A, D), union(B, E), union(C, F):

\[
\begin{array}{cccccccc}
D^1 & E^1 & F^1 & G^0 \\
A^0 & B^0 & C^0 \\
\end{array}
\]
Disjoint Sets with Union by Depth (3)

After union($C, G$), union($E, A$):

After union($B, G$):
Complexity of disjoint sets w/ union by depth

- Asymptotic complexity of `makeSet` unchanged.
- `union` has become a bit more expensive, but only modestly.
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- What about find?
  - A sequence of $N$ operations can create at most $N$ elements
  - So, maximum set size is $O(N)$
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  - With union by rank, each increase in rank can occur only after a doubling of elements in the set
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Observation

The number of nodes of rank $k$ never exceeds $N/2^k$
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- What about find?
  - A sequence of $N$ operations can create at most $N$ elements
  - So, maximum set size is $O(N)$
  - With union by rank, each increase in rank can occur only after a doubling of elements in the set

Observation

The number of nodes of rank $k$ never exceeds $N/2^k$

- So, height of trees is bounded by $\log N$
Complexity of disjoint sets w/ union by depth (2)

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  - **Question:** Is this bound tight?
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Complexity of disjoint sets w/ union by depth (2)

- Height of trees is bounded by \( \log N \)
- Thus we have a complexity of \( O(\log N) \) for \( \text{find} \)
  - \textbf{Question}: Is this bound tight?

\begin{quote}
From here on, we limit union operations to only root nodes, so their cost is \( O(1) \).

This requires \text{find} to be moved out of union into a separate operation, and hence the total number of operations increases, but only by a constant factor.
\end{quote}
Improving \texttt{find} performance

**Idea:** Why not force depth to be 1? Then \texttt{find} will have $O(1)$ complexity!
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**Approach:** **Threaded Trees**

---

\texttt{find}:

\begin{verbatim}
int find(int x)
{ return leader(x); }
\end{verbatim}

\texttt{MAKE SET}:

\begin{verbatim}
void MAKE SET(int x)
{ leader(x) = x; next(x) = x; size(x) = 1; }
\end{verbatim}

\texttt{UNION}:

\begin{verbatim}
void UNION(int x, int y)
{ x = FIND(x); y = FIND(y);
  if (size(x) > size(y))
    UNION(y, x);
  else
    UNION(x, y);
  size(leader(y)) += size(y); }
\end{verbatim}

---

**WORST CASE SEQUENCE** \((n)\):

\begin{verbatim}
for (i = 2; i <= n; i++)
  MAKE SET(i);
  UNION(1, i);
\end{verbatim}

---

We are being stupid in two different ways here. One is the order of operations in WORST CASE SEQUENCE. Obviously, it would be more efficient to merge the sets in the other order, or to use some sort of divide and conquer approach. Unfortunately, we can’t fix this; we don’t get to decide how our data structures are used! The other is that we always update the leader pointers in the larger set. To fix this, we add a comparison inside the UNION algorithm to determine which set is smaller. This requires us to maintain the size of each set, but that’s easy.

**MAKE WEIGHTED SET**:

\begin{verbatim}
void MAKE WEIGHTED SET(int x)
{ leader(x) = x; next(x) = x; size(x) = 1; }
\end{verbatim}

**WEIGHTED UNION**:

\begin{verbatim}
void WEIGHTED UNION(int x, int y)
{ x = FIND(x); y = FIND(y);
  if (size(x) > size(y))
    UNION(y, x);
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    UNION(x, y);
  size(leader(y)) += size(y); }
\end{verbatim}

---

The new WEIGHTED UNION algorithm still takes \(\Theta(n)\) time to merge two \(n\)-element sets. However, in an amortized sense, this algorithm is much more efficient. Intuitively, before we can merge two large sets, we have to perform a large number of MAKE WEIGHTED SET operations.
Improving `find` performance

**Idea:** Why not force depth to be 1? Then `find` will have $O(1)$ complexity!

**Approach:** Threaded Trees

The worst-case running time of `union` is a constant times the size of the larger set. Thus, if we merge a one-element set with another $n$-element set, the running time can be $\Theta(n)$. Generalizing this idea, it is quite easy to come up with a sequence of $n$ `MAKE SET` and $n-1$ `UNION` operations that requires $\Theta(n^2)$ time to create the set \{1, 2, ..., $n$\} from scratch.

**Worst Case Sequence ($n$):**

```
MAKE SET(1)
for i ← 2 to n
  MAKE SET(i)
  UNION(1, i)
```

We are being stupid in two different ways here. One is the order of operations in `Worst Case Sequence`. Obviously, it would be more efficient to merge the sets in the other order, or to use some sort of divide and conquer approach. Unfortunately, we can't fix this; we don't get to decide how our data structures are used! The other is that we always update the leader pointers in the larger set. To fix this, we add a comparison inside the `UNION` algorithm to determine which set is smaller. This requires us to maintain the size of each set, but that's easy.

**Solution:**
- Merge smaller set with larger set
- Amortize cost of `union` over other operations
Sets w/ threaded trees: Amortized analysis

- Other than cost of updating parent pointers, union costs $O(1)$
Sets w/ threaded trees: Amortized analysis

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- **Key observation:** Each time an element’s parent pointer changes, it is in a set that is twice as large as before
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  - So, with $n$ operations, you can at most $O(\log n)$ parent pointer updates per element

- Thus, amortized cost of $n$ operations, consisting of some mix of `makeset`, `find` and union is at most $n \log n$
Further improvement

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  - Actually, we promote *x*, π(*x*), π(π(*x*)), π(π(π(*x*)))* and so on.
  - As a result, subsequent calls to find *x* or its parents become cheap.

- From here on, we let *rank* be defined by the *union* algorithm
  - For root node, *rank* is same as depth
  - But once a node becomes a non-root, its *rank* stays fixed,
    - *even when path compression decreases its depth.*
little more careful to maintain our data structure in good shape. As any housekeeper knows, a little extra effort put into routine maintenance can pay off handsomely in the long run, by forestalling major calamities. We have in mind a particular maintenance operation for our union-find data structure, intended to keep the trees short— during each find, when a series of parent pointers is followed up to the root of a tree, we will change all these pointers so that they point directly to the root (Figure 5.7). This path compression heuristic only slightly increases the time needed for a find and is easy to code.

\[
\text{function find}(x) \\
\text{if } x \neq \pi(x) : \\
\pi(x) = \text{find}(\pi(x)) \\
\text{return } \pi(x)
\]

The benefit of this simple alteration is long-term rather than instantaneous and thus necessitates a particular kind of analysis: we need to look at sequences of find and union operations, starting from an empty data structure, and determine the average time per operation. This amortized cost turns out to be just barely more than $O(1)$, down from the earlier $O(\log n)$.
Sets w/ Path compression: Amortized analysis

Amortized cost per operation of $n$ set operations is $O(\log^* n)$ where

$$\log^* x = \text{smallest } k \text{ such that } \underbrace{\log(\cdots \log(x) \cdots)}_{k \text{ times}} = 1$$
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*Note:* \( \log^*(x) \leq 5 \) for virtually any \( n \) of practical relevance. Specifically,

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\log^*(2^{65536}) = \log^*(2^{2^{2^2}}) = 5
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Note that \( 2^{65536} \) is approximately a 20,000 digit decimal number.

We will never be able to store input of that size, at least not in our universe. (Universe contains may be \( 10^{100} \) elementary particles.)

So, we might as well treat \( \log^*(n) \) as \( O(1) \).
For $n$ operations, rank of any node falls in the range $[0, \log n]$
Path compression: Amortized analysis (2)

- For $n$ operations, rank of any node falls in the range $[0, \log n]$

- Divide this range into following groups:

  $[1], [2], [3-4], [5-16], [17-2^{16}], [2^{16} + 1-2^{65536}], \ldots$

  Each range is of the form $[k-2^{k-1}]$
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- Note: when a node becomes a non-root, its rank never changes
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**Key Idea**

Give an “allowance” to a node when it becomes a non-root. This allowance will be used to pay costs of path compression operations involving this node.
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**Key Idea**

Give an “allowance” to a node when it becomes a non-root. This allowance will be used to pay costs of path compression operations involving this node.

For a node whose rank is in the range \([k–2^{k-1}]\), the allowance is \(2^{k-1}\).
Total allowance handed out

- Recall that number of nodes of rank \( r \) is at most \( n/2^r \)
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- Recall that number of nodes of rank $r$ is at most $n/2^r$
- Recall that a node of rank is in the range $[k-2^{k-1}]$ is given an allowance of $2^{k-1}$. 
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- Recall that number of nodes of rank $r$ is at most $n/2^r$
- Recall that a node of rank is in the range $[k−2^{k−1}]$ is given an allowance of $2^{k−1}$.
- Total allowance handed out to nodes with ranks in the range $[k−2^{k−1}]$ is therefore given by

$$2^{k−1} \left( \frac{n}{2^k} + \frac{n}{2^{k+1}} + \cdots + \frac{n}{2^{2^{k−1}}} \right) \leq 2^{k−1} \frac{n}{2^{2^{k−1}}} = n$$
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- Since total number of ranges is $\log^* n$, total allowance granted to all nodes is $n \log^* n$
- We will spread this cost across all $n$ operations, thus contributing $O(\log^* n)$ to each operation.
Paying for all find’s

- Cost of a find equals # of parent pointers followed
  - Each pointer followed is updated to point to root of current set.

Key idea: Charge the cost of updating $\pi(p)$ to:

Case 1: If $G(\pi(p)) \neq G(p)$, then charge it to the current find operation.

Can apply only $\log^* n$ times: a leaf’s $G$-value is at least 1, and the root’s $G$-value is at most $\log^* n$.

Case 2: Otherwise, charge it to $p$’s allowance.

Need to show that we have enough allowance to pay each time this case occurs.
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  - Adds only $\log^* n$ to cost of $\texttt{find}$
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    - Need to show that we have enough allowance to to pay each time this case occurs.
Paying for all `find`’s (2)

- If $\pi(p)$ is updated, then the rank of $p$’s parent increases.
Paying for all \texttt{find}'s (2)

- If $\pi(p)$ is updated, then the rank of $p$'s parent increases.

- Let $p$ be involved in a series of \texttt{find}'s, with $q_i$ being its parent after the $i$th \texttt{find}. Note

\[
\text{rank}(p) < \text{rank}(q_0) < \text{rank}(q_1) < \text{rank}(q_2) < \cdots
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- Let $m$ be the number of such operations before $p$’s parent has a higher $G$-value than $p$, i.e.,

  $$G(p) = G(q_m) < G(q_{m+1}).$$
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- Recall that
  - A \( G(p) = r \) then \( r \) corresponds to a range \([k−2^{k−1}]\) where \( k \leq \text{rank}(p) \leq 2^{k−1} \). Since
    \[ G(p) = G(q_m), \ q_m \leq 2^{k−1} \]
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  - The allowance given to $p$ is also $2^{k-1}$
- So, there is enough allowance for all promotions up to $m$.
- After $m+1$th find, the find operation will pay for pointer updates, as $G(\pi(p)) > G(p)$ from here on.