A relation $R$ from a set to itself is often more interesting than a relation from one set to another.

- $R$ can be composed with itself
- can be represented using a *directed graph* or *digraph*. 

Relations on a Set (Textbook Ch. 9)
A Digraph $G = (V, E)$ where

- $V$ is the set of *vertices* or *nodes*, and
- $E$ is a set of directed edges of the form $(u, v)$ where $u, v \in V$. 
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  - An edge with the same head and tail is called a self-loop.

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$G$ is nothing but a relation from $V$ to $V$

In fact, we have already called the “arrows” in visual representation of relations as graphs!
**Degree**

Degree: number of arrows coming into (“in” degree) or the number of arrows going out (“out” degree)

**Property of Degrees in a Graph**

\[
\sum_{v \in V(G)} \text{indeg}(v) = \sum_{v \in V} \text{outdeg}(v)
\]
A walk in a graph $G$ is a sequence of vertices $v_1, v_2, \ldots, v_n$ such that

- Every $v_i \in V(G)$, and
- $(v_j, v_{j+1}) \in E(G)$
- We say that the walk starts at $v_1$ and ends at $v_n$
- The length of the walk is $n - 1$
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A closed walk is a walk with $v_1 = v_n$.

A path is a walk where all the vertices are distinct.

A cycle is a closed walk where $v_1, \ldots, v_{n-1}$ form a path.
Some Properties of Walks

Theorem

- The shortest walk between two vertices $u$ and $v$ is a path.
- The shortest closed walk through a vertex $v$ is a cycle.

$\text{dist}(u, v)$: is the shortest path between $u$ and $v$. 

The Triangle Inequality

$\text{dist}(u, v) \leq \text{dist}(u, w) + \text{dist}(w, v)$
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The Triangle Inequality

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Euler and Hamiltonian Tours

**Euler Tour:** A closed walk that visits every edge in the graph exactly once.

- See Wikipedia for examples and history. (Look only at the paragraphs before table of content.)

**Hamiltonian Tour:** A cycle that visits every vertex exactly once.

- Also known as *Traveling Salesman Problem*
Euler Tour: Necessary and Sufficient Condition

\[ \forall v \in V \; \text{indeg}(v) = \text{outdeg}(v) \]
Think of $E$ as defining a relation from $V$ to $V$

- What do walks of length 2 denote?
- What about walks of length $n$?
Properties of a Relation $R : V \rightarrow V$

Reflexive: $\forall a \ aRa$
- Graph has self-loops at every vertex

Irreflexive: $\forall a \ a \not\in R a$
- No self-loops
Properties of a Relation $R : V \rightarrow V$

**Symmetric:** $\forall a, b \ aRb \rightarrow bRa$

- Edges come in pairs: we can merge them into one and remove arrows, leading to undirected graphs

**Anti-symmetric:** $\forall a, b \ aRb \rightarrow (a = b \ \lor \ b \not{R} a)$
Properties of a Relation $R : V \rightarrow V$

**Transitive:** $\forall a, b, c \ aRb \land bRc \rightarrow aRc$

- Any vertex $b$ reachable from $a$ is reachable in a single step.
Properties of a Relation $R : V \rightarrow V$

Reflexive: $\forall a \ aRa$

Irreflexive: $\forall a \ a \not\in R a$

Symmetric: $\forall a, b \ aRb \rightarrow bRa$

Anti-symmetric: $\forall a, b \ aRb \rightarrow (a = b \lor b \not\in R a)$

Transitive: $\forall a, b, c \ aRb \land bRc \rightarrow aRc$
Closure Operations

Start with a relation, introduce additional edges implied by properties discussed before:

- Reflexive closure
- Symmetric closure
- Transitive closure
Reflexive Closure

- Add self-loops at every vertex
Symmetric Closure

- Add edge from $b$ to $a$ whenever there is an edge from $a$ to $b$
Transitive Closure

- Add edge from $a$ to $c$ iff there is an edge from $a$ to $b$ and another edge from $b$ to $c$. 
Partial Orders: Anti-symmetric and transitive. Forms Directed Acyclic Graphs (DAGs)

Linear order: Partial order where every pair of elements is comparable

i.e., either $aRb$ or $bRa$ holds.
DAGs, Dependencies, Topological Sort and Scheduling ...

See Textbook Section 9.5
Properties of a Relations: Equivalence Relations

- Reflexive, Symmetric and Transitive
- Partition a domain into *Equivalence Classes*

\[ EC(a) = \{ b | aRb \} \]

- Examples
  - \( aRb \) iff \( a, b \in \mathbb{N}, a \mod n = b \mod n \)
  - Connected components of a graph