Relations on a Set (Textbook Ch. 9)

- A relation R from a set to itself is often more interesting than a relation from one set to another.
  - R can be composed with itself
  - can be represented using a *directed graph* or *digraph*.

Digraph

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  - V is the set of *vertices* or *nodes*, and
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- G is nothing but a relation from V to V
- In fact, we have already called the "arrows" in visual representation of relations as graphs!

Degree

Degree: number of arrows coming into ("in" degree) or the number of arrows going out ("out" degree)

Property of Degrees in a Graph

$$\sum_{v \in V(G)} indeg(v) = \sum_{v \in V} outdeg(v)$$

Walks and Paths

- A walk in a graph G is a sequence of vertices $v_1, v_2, \ldots, v_n$ such that
  - Every $v_i \in V(G)$, and
  - $(v_j, v_{j+1}) \in E(G)$
  - We say that the walk starts at v_1 and ends at v_n
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  - The *length* of the walk is $n - 1$
- A *closed walk* is a walk with $v_1 = v_n$.
- A *path* is a walk where all the vertices are distinct.
- A *cycle* is a closed walk where $v_1, \ldots, v_{n-1}$ form a path.

Walks and Paths

Some Properties of Walks

Theorem

- The shortest walk between two vertices u and v is a path.
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The Triangle Inequality

$$dist(u, v) \leq dist(u, w) + dist(w, v)$$

Euler and Hamiltonian Tours

Euler Tour: A closed walk that visits every edge in the graph exactly once.

- See Wikipedia for examples and history. (Look only at the paragraphs before table of content.)

Hamiltonian Tour: A cycle that visits every vertex exactly once.

- Also known as *Traveling Salesman Problem*

Euler Tour: Necessary and Sufficient Condition

$$\forall v \in V \;\; indeg(v) = outdeg(v)$$

Thinking About Relations Using Graphs

- Think of E as defining a relation from V to V
  - What do walks of length 2 denote?
  - What about walks of length n?

Properties of a Relation $R : V \longrightarrow V$

Reflexive: $\forall a \;\; aRa$

- Graph has self-loops at every vertex

Irreflexive: $\forall a \;\; a \not R a$

- No self-loops

Properties of a Relation $R : V \longrightarrow V$

Symmetric: $\forall a, b \;\; aRb \rightarrow bRa$

- Edges come in pairs: we can merge them into one and remove arrows, leading to undirected graphs

Anti-symmetric: $\forall a, b \;\; aRb \rightarrow (a = b \;\; \vee \;\; b \not R a)$

Properties of a Relation $R : V \longrightarrow V$

Transitive: $\forall a, b, c \;\; aRb \wedge bRc \rightarrow aRc$

- Any vertex b reachable from a is reachable in a single step.

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Closure Operations

- Start with a relation, introduce additional edges implied by properties discussed before:
  - Reflexive closure
  - Symmetric closure
  - Transitive closure

Reflexive Closure

- Add self-loops at every vertex

Symmetric Closure

- Add edge from b to a whenever there is an edge from a to b

Transitive Closure

- Add edge from a to c iff there is an edge from a to b and another edge from b to c.

Properties of a Relations

Partial Orders: Anti-symmetric and transitive. Forms Directed Acyclic Graphs (DAGs)

Linear order: Partial order where every pair of elements is comparable

- i.e., either aRb or bRa holds.

DAGs, Dependencies, Topological Sort and Scheduling ...

See Textbook Section 9.5

Properties of a Relations: Equivalence Relations

- Reflexive, Symmetric and Transitive
- Partition a domain into *Equivalence Classes*

$$EC(a) = \{b | aRb\}$$

- Examples
  - $aRb \;\; iff \;\; a, b \in \mathbb{N}, \; a \; mod \; n = b \; mod \; n$
  - Connected components of a graph