Sets (Textbook §4.1)

Informally, a set is a collection of things:

$$
\begin{aligned}
A &= \{\mathit{Alex}, \mathit{Tippy}, \mathit{Shells}, \mathit{Shadow}\} && \text{Pet names} \\
B &= \{\mathit{red}, \mathit{blue}, \mathit{yellow}\} && \text{Primary colors} \\
C &= \{\{a, b\}, \{b, c\}, \{a, d\}, \{a, b, c\}\} && \text{A set of sets}
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- The order of elements is irrelevant — a set is an unordered collection
- There are no repeated elements in a set

Infinite Sets and Set Builder Notation

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or more compactly as:

$$T ::= \{3n | n \in \mathbb{N}\}$$

Predefined Sets

$\emptyset$	The empty set
$\mathbb{N}$	The set of natural numbers, i.e., $\{0, 1, 2, 3, \ldots\}$
$\mathbb{Z}$	The set of integers, i.e., $\{0, -1, 1, -2, 2, \ldots\}$
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of Complex numbers

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Proper Subset	$\subset$	$A \subset B$ iff $A \subseteq B$ and $A \neq B$

Showing Two Sets A and B are Equal

- Show that $x \in A$ implies $x \in B$ and vice-versa
- Show $A \subseteq B$ and $B \subseteq A$

Universal Set and Complement

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- In the presence of a universal set, we can define a *complement* $\overline{A}$ as follows:

$$\overline{A} = U - A$$

Alternatively:

$$\overline{A} = \{x \in U | x \notin A\}$$

Some properties of Set Operators

- $\cup$ and $\cap$ are commutative and associative
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$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
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- De Morgan's Law

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Power Set

Given a set A, the *power set* of A, denoted $\wp(A)$ is

$$\wp(A) = \{x | x \subseteq A\}$$

- A powerset always includes $\emptyset$
- $\wp(A)$ always includes A

Size of Power Set

If the size of A, denoted $|A|$, is n, what is the size of $\wp(A)$?

Products and Tuples

Cartesian Product of sets A and B: $A \times B = \{(a, b) | a \in A, b \in B\}$

Example: Let $A = \{1, 4, 9\}$, $B = \{a, e, i, o, u\}$

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$$\begin{aligned} A \times B = \{ \; & (1, a), (1, e), (1, i), (1, o), (1, u), \\ & (4, a), (4, e), (4, i), (4, o), (4, u), \\ & (9, a), (9, e), (9, i), (9, o), (9, u)\} \end{aligned}$$

Products and Tuples

- (x, y) is called an (ordered) *pair*
- (x, y, z) is a *triple*
- More generally, $(x_1, x_2, \ldots, x_n)$ is an n-tuple, or just a *tuple*
- *Unlike sets, the order of components is important in a tuple.* $A \times B \neq B \times A$
- A^n is short hand for

$$\underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$$

Sets: Summary

- Definition of sets
- Set builder notation
- Set operators: membership, subset, union, intersection, difference
  - Properties of set operators: commutativity, associativity, distributivity
- Set equality
- Universal set, set complement and De Morgan's Laws
- Power Set
- Cartesian Product and Tuples