Functions (Textbook §4.3)

A *function* associates each element of a set A with a ***unique*** element of another set B.

$$f\colon A \longrightarrow B$$

Example: $f(x) = x^2$ defines a function from $f\colon \mathbb{N} \longrightarrow \mathbb{N}$.

Pictorial representation:

Functions: Terminology

- For a function $f\colon A \longrightarrow B$, the set A is called the *domain*, while B is called *codomain*.
- Elements of A with an outgoing arrow are called the *support* of f.
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- For a set $S \subseteq A$, we define $f(s) = \{f(x) | x \in S\}$. This set is called the *image* of S.

Functions: More Examples

$$f\colon \mathbb{N} \longrightarrow \mathbb{N} \text{ where } f(x) = \sqrt{x}$$

- What is the domain?
- What is the codomain?

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- What is its support set?
- What is the range?

Functions: More Examples

$$f\colon \mathbb{N} \longrightarrow \mathbb{N} \text{ where } f(x) = \sqrt{x}$$

- Is f total or partial?
- What is the image of $\{9, 36, 4, 324, 1024\}$?

Function Composition

Given $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$, their *composition*, denoted $g \circ f$ is given by:

$$(g \circ f)(x) = g(f(x))$$

Examples:

- $f(x) = 2x, \; g(x) = 3x$
- $f(x) = x^2, \; g(y) = \sqrt{y}$

Functions with Multiple Arguments

Example: $f(x, y) = x + y$

- Instead of saying f takes two arguments, we say it takes one argument that is a pair.

$$f\colon \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

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- $f(p, q, r) = p \wedge (q \vee r)$ is $f\colon B \times B \times B \longrightarrow B$
  - Here, B stands for the set $\{\mathbf{T}, \mathbf{F}\}$ of boolean values.

Binary Relations

- Like functions, relations associate elements of set A with elements of set B.
- Unlike a function, the same element $a \in A$ may be associated with multiple elements of B.

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- Relations are typically specified using a predicate.
- Like functions, relations can represent associations between multiple sets, but we are most interested in *binary* relations.
  - We treat n-ary relations as binary relations over product sets

Binary Relations: Terminology

A relation $R\colon A \longrightarrow B$ is said to be "between A and B."

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Graph: A subset of $A \times B$ that consists of all a, b such that $a \; R \; b$.

- This graph can be visualized using an arrow from a to b whenever $a \; R \; b$.

Examples of Graphs (Representing Relations)

Examples of Graphs (Representing Functions)

Relational Composition and Inverse

Composition: For $R\colon A \longrightarrow B$ and $S\colon B \longrightarrow C$,

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- But, the inverse of a function f may not always be a function
  - f^{-1} is a function iff f is injective

Classifying Relations Based on its Graph

function: if it has [≤ 1 arrow **out**] property

total: if it has [≥ 1 arrow **out**] property

surjective: if it has [≥ 1 arrow **in**] property

injective: if it has [≤ 1 arrow **in**] property

bijective: if it has *all of the above properties*
i.e., it has [$= 1$ arrow **out**] and [$= 1$ arrow **in**].

Using Injection and Surjection to Relate Set Cardinalities

A surj B iff there is a *surjective* function from A to B

A inj B iff there is a *injective, total* function from A to B

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For *finite* sets

- $|A| \geq |B|$ iff A *surj* B
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Counting Using Bijections: Power Set Size Revisited

Counting Infinite Sets (Textbook §8.1)

Can we use the same ideas as finite sets?

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Basically. But:

- There are some unintuitive things about the "size" of infinite sets
- We don't know how to say one set is stricly larger
- We don't know how to measure the size of an infinite set.

We will ignore the third problem, and just talk about comparing sizes.

Infinite Sets are Different ...

For finite sets, adding an element strictly increases its size

- i.e., if A is finite, and $b \notin A$, there is no bijection from A to $A \cup \{b\}$

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This is not true for infinite sets In fact:

A set A is infinite iff there is a bijection from A to $A \cup \{b\}$