A function associates each element of a set $A$ with a unique element of another set $B$.

\[ f: A \longrightarrow B \]

**Example:** $f(x) = x^2$ defines a function from $f: \mathbb{N} \longrightarrow \mathbb{N}$.

Pictorial representation:
Functions: Terminology

- For a function $f: A \rightarrow B$, the set $A$ is called the *domain*, while $B$ is called *codomain*.

- Elements of $A$ with an outgoing arrow are called the *support* of $f$.

- Elements of $B$ that have an incoming arrow are called the *range*.
For a function $f: A \rightarrow B$, the set $A$ is called the \textit{domain}, while $B$ is called \textit{codomain}.

Elements of $A$ with an outgoing arrow are called the \textit{support} of $f$.

Elements of $B$ that have an incoming arrow are called the \textit{range}.

\textit{Partial function}: Value of $f$ is undefined for some arguments. In other words, the support set is not the entire domain.

\textit{Total function}: $f(x)$ is defined $\forall x \in A$. In other words, the support set is $A$ itself.
Functions: Terminology

- For a function $f: A \rightarrow B$, the set $A$ is called the **domain**, while $B$ is called **codomain**.

- Elements of $A$ with an outgoing arrow are called the **support** of $f$.

- Elements of $B$ that have an incoming arrow are called the **range**.

- **Partial function**: Value of $f$ is undefined for some arguments. In other words, the support set is not the entire domain.

- **Total function**: $f(x)$ is defined $\forall x \in A$. In other words, the support set is $A$ itself.

- For a set $S \subseteq A$, we define $f(s) = \{f(x)|x \in S\}$. This set is called the **image** of $S$. 
Functions: More Examples

\[ f: \mathbb{N} \rightarrow \mathbb{N} \text{ where } f(x) = \sqrt{x} \]

- What is the domain?
- What is the codomain?
Functions: More Examples

\[ f: \mathbb{N} \to \mathbb{N} \text{ where } f(x) = \sqrt{x} \]

- What is its support set?
- What is the range?
Functions: More Examples

\[ f: \mathbb{N} \longrightarrow \mathbb{N} \text{ where } f(x) = \sqrt{x} \]

- Is \( f \) total or partial?
- What is the image of \( \{9, 36, 4, 324, 1024\} \)?
Function Composition

Given $f: A \rightarrow B$ and $g: B \rightarrow C$, their \textit{composition}, denoted $g \circ f$ is given by:

$$(g \circ f)(x) = g(f(x))$$

Examples:

- $f(x) = 2x$, $g(x) = 3x$
- $f(x) = x^2$, $g(y) = \sqrt{y}$
Functions with Multiple Arguments

Example: $f(x, y) = x + y$

- Instead of saying $f$ takes two arguments, we say it takes one argument that is a pair.

$$f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$$
Functions with Multiple Arguments

Example: \( f(x, y) = x + y \)

- Instead of saying \( f \) takes two arguments, we say it takes one argument that is a pair.
  \[ f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \]

- We can extend to any number of arguments:
  \[ g: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \]
  is a function that takes 4-tuple argument (all real numbers) and returns one value, as given by \( f(x, y, z, u) = x^2 + y^2 + z^2 + u^2 \).
Functions with Multiple Arguments

Example: \( f(x, y) = x + y \)

- Instead of saying \( f \) takes two arguments, we say it takes one argument that is a pair.
  \[
  f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}
  \]

- We can extend to any number of arguments:
  \[
  g: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}
  \]
  is a function that takes 4-tuple argument (all real numbers) and returns one value, as given by 
  \[
  f(x, y, z, u) = x^2 + y^2 + z^2 + u^2.
  \]

- \( f(p, q, r) = p \wedge (q \lor r) \) is \( f: B \times B \times B \rightarrow B \)
  - Here, \( B \) stands for the set \( \{T, F\} \) of boolean values.
Binary Relations

- Like functions, relations associate elements of set $A$ with elements of set $B$.
- Unlike a function, the same element $a \in A$ may be associated with multiple elements of $B$.

**Example:** $\leq : \mathbb{N} \rightarrow \mathbb{N}$
Binary Relations

- Like functions, relations associate elements of set $A$ with elements of set $B$.
- Unlike a function, the same element $a \in A$ may be associated with multiple elements of $B$.

**Example:** $\leq : \mathbb{N} \rightarrow \mathbb{N}$

- Relations are typically specified using a predicate.
- Like functions, relations can represent associations between multiple sets, but we are most interested in *binary* relations.
- We treat $n$-ary relations as binary relations over product sets.
A relation $R: A \rightarrow B$ is said to be “between $A$ and $B$.”

If $A = B$, we say $R$ is a relation on $A$. 
A relation $R: A \longrightarrow B$ is said to be “between $A$ and $B$.”

If $A = B$, we say $R$ is a relation on $A$.

**Domain:** the set $A$

**Codomain:** the set $B$

**$a R b$:** denotes that $a \in A$ and $b \in B$ are related by $R$. 
A relation $R : A \rightarrow B$ is said to be “between $A$ and $B$.”
If $A = B$, we say $R$ is a relation on $A$.

Domain: the set $A$

Codomain: the set $B$

$a R b$: denotes that $a \in A$ and $b \in B$ are related by $R$.

Image of set $C \subseteq A$: Similar to images for functions:

$$\{ b | \ b \in B \text{ and } \exists c \in C \ c R b \}$$
A relation $R: A \rightarrow B$ is said to be “between $A$ and $B$.” If $A = B$, we say $R$ is a relation on $A$.

**Domain:** the set $A$

**Codomain:** the set $B$

**$a R b$:** denotes that $a \in A$ and $b \in B$ are related by $R$.

**Image of set $C \subseteq A$:** Similar to images for functions:

$$\{ b \mid b \in B \text{ and } \exists c \in C \ c R b \}$$

**Graph:** A subset of $A \times B$ that consists of all $a, b$ such that $a R b$.

- This graph can be visualized using an arrow from $a$ to $b$ whenever $a R b$. 
Examples of Graphs (Representing Relations)
Examples of Graphs (Representing Functions)
Relational Composition and Inverse

Composition: For \( R: A \to B \) and \( S: B \to C \),

\[
a (R \circ S) \, c := \exists b \in B \ a \, R \, b \land b \, S \, c
\]
Relational Composition and Inverse

Composition: For $R: A \rightarrow B$ and $S: B \rightarrow C$,

$$a (R \circ S) c ::= \exists b \in B \ a R b \land b S c$$

Inverse: The inverse of a relation $R$, denoted $R^{-1}$, is given by

$$b R^{-1} a \iff a R b$$

- Inverse of functions can be defined in the same way.
- But, the inverse of a function $f$ may not always be a function
Relational Composition and Inverse

Composition: For $R: A \rightarrow B$ and $S: B \rightarrow C$,

$$a (R \circ S) c ::= \exists b \in B \ a R b \land b S c$$

Inverse: The inverse of a relation $R$, denoted $R^{-1}$, is given by

$$b R^{-1} a \iff a R b$$

- Inverse of functions can be defined in the same way.
- But, the inverse of a function $f$ may not always be a function
  - $f^{-1}$ is a function iff $f$ is injective
Classifying Relations Based on its Graph

function: if it has $\leq 1$ arrow \textbf{out} property

total: if it has $\geq 1$ arrow \textbf{out} property

surjective: if it has $\geq 1$ arrow \textbf{in} property

injective: if it has $\leq 1$ arrow \textbf{in} property

bijective: if it has \textit{all of the above properties} \\
   i.e., it has $= 1$ arrow \textbf{out} and $= 1$ arrow \textbf{in}.
Using Injection and Surjection to Relate Set Cardinalities

\[
\begin{align*}
A \text{ surj } B & \iff \text{there is a \textit{surjective} function from } A \text{ to } B \\
A \text{ inj } B & \iff \text{there is a \textit{injective, total} function from } A \text{ to } B \\
A \text{ bij } B & \iff \text{there is a \textit{bijection} from } A \text{ to } B
\end{align*}
\]
Using Injection and Surjection to Relate Set Cardinalities

\[ A \text{ surj } B \text{ iff there is a surjective function from } A \text{ to } B \]

\[ A \text{ inj } B \text{ iff there is a injective, total function from } A \text{ to } B \]

\[ A \text{ bij } B \text{ iff there is a bijection from } A \text{ to } B \]

For \textit{finite} sets

\[ |A| \geq |B| \text{ iff } A \text{ surj } B \]

\[ |A| \leq |B| \text{ iff } A \text{ inj } B \]

\[ |A| = |B| \text{ iff } A \text{ bij } B \]
Counting Using Bijections: Power Set Size Revisited
Counting Infinite Sets (Textbook §8.1)

Can we use the same ideas as finite sets?

- $|A| \geq |B|$ iff $A$ surj $B$
- $|A| \leq |B|$ iff $A$ inj $B$
- $|A| = |B|$ iff $A$ bij $B$
Can we use the same ideas as finite sets?

- $|A| \geq |B|$ iff $A$ surj $B$
- $|A| \leq |B|$ iff $A$ inj $B$
- $|A| = |B|$ iff $A$ bij $B$

Basically. But:

- There are some unintuitive things about the “size” of infinite sets
- We don’t know how to say one set is strictly larger
- We don’t know how to measure the size of an infinite set.

We will ignore the third problem, and just talk about comparing sizes.
Infinite Sets are Different ...

For finite sets, adding an element strictly increases its size

- i.e., if $A$ is finite, and $b \notin A$, there is no bijection from $A$ to $A \cup \{b\}$
Infinite Sets are Different ...

For finite sets, adding an element strictly increases its size

- i.e., if $A$ is finite, and $b \not\in A$, there is no bijection from $A$ to $A \cup \{b\}$

This is not true for infinite sets
Infinite Sets are Different ...

For finite sets, adding an element strictly increases its size

- i.e., if $A$ is finite, and $b \notin A$, there is no bijection from $A$ to $A \cup \{b\}$

This is not true for infinite sets. In fact:

A set $A$ is infinite iff there is a bijection from $A$ to $A \cup \{b\}$