Recurrences and Recursion

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Intro

- Recursion is one of the most versatile techniques in programming as well as algorithm design
- Closely related to induction:
  - Consists of a base case and recursive case, similar to base and inductive steps.
  - Correctness of recursive algorithms is proved by induction
- Example: Computing Fibonacci numbers:

  Base case(s): $F(0) = 0, F(1) = 1$

  Recursive case: $F(n) = F(n-1) + F(n-2)$

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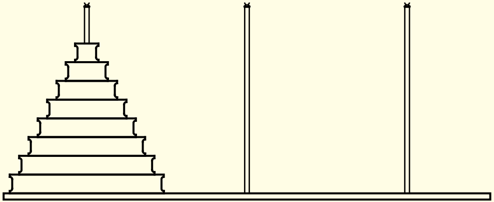
Recursive data types: e.g., lists, trees,...

Base Case: empty list (for lists), leaf (for trees),...

Recursive Case: pair of element and rest of list, pair of trees, etc.

Recurrences: Tower of Hanoi Problem



Goal: Move all disks from one post to another.

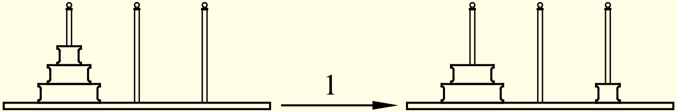
Rules:

- Only the top-most disk can be moved.
- No disk can be placed on a smaller disk.

Questions:

- How do you solve the puzzle?
- How many moves will be needed?

Tower of Hanoi Problem: Example with Three Disks



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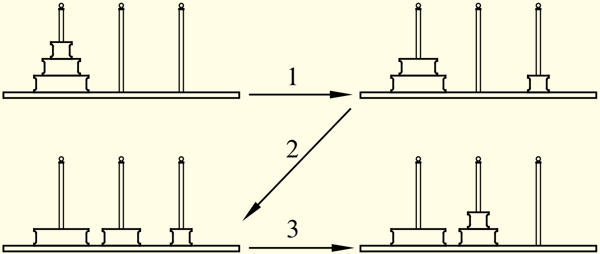

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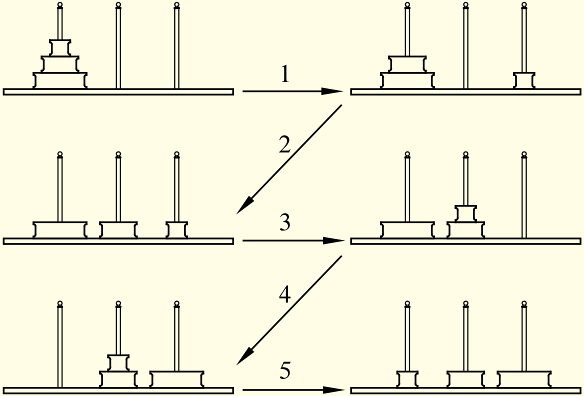

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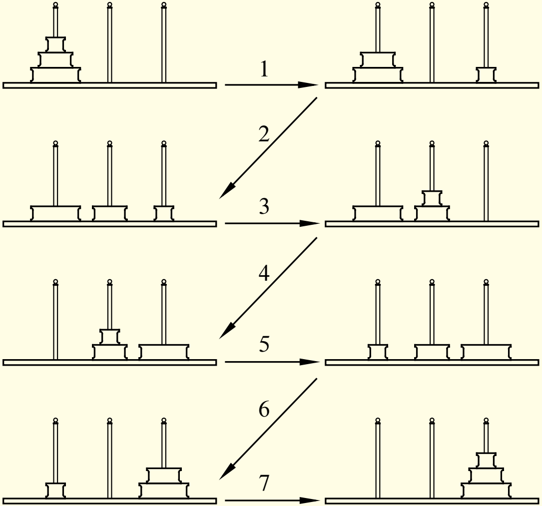
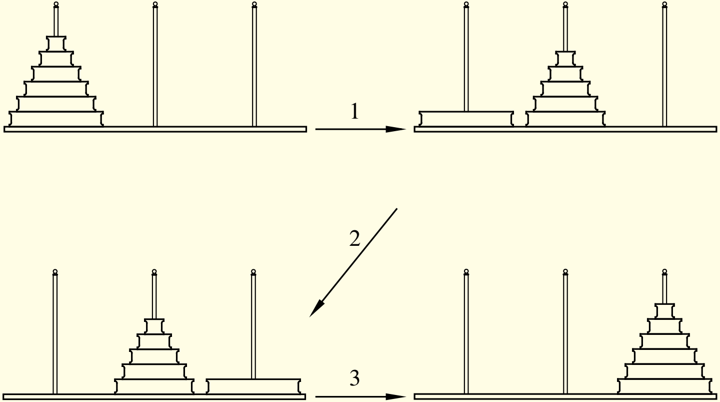


A Recursive Algorithm for Tower of Hanoi Problem



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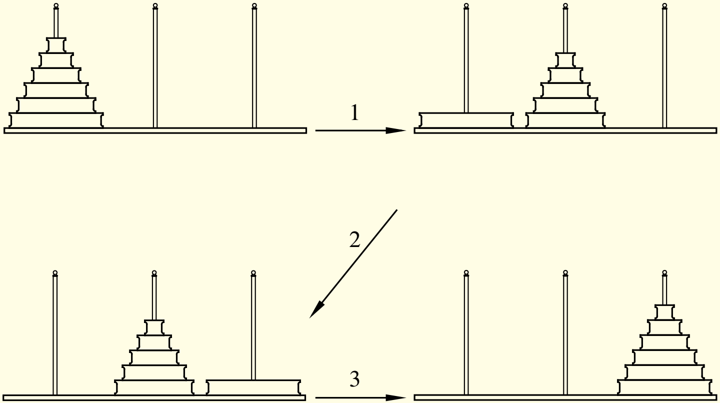

MoveStack$(n, 1, 3)$:

- *MoveStack*$(n-1, 1, 2)$
- *MoveDisk*$(n, 1, 3)$
- *MoveStack*$(n-1, 2, 3)$

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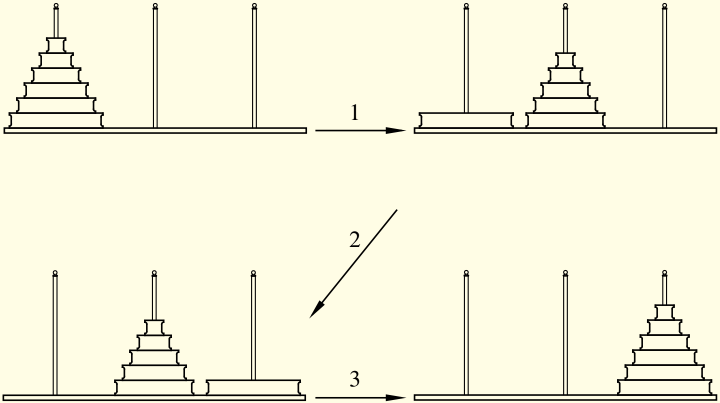

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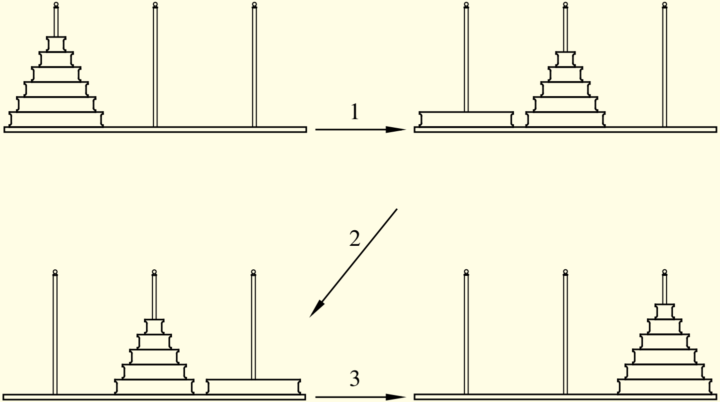
Base Case:

MoveStack$(1, x, y)$

- *MoveDisk*$(1, x, y)$

A Recurrence for the Runtime of Towers of Hanoi Algorithm



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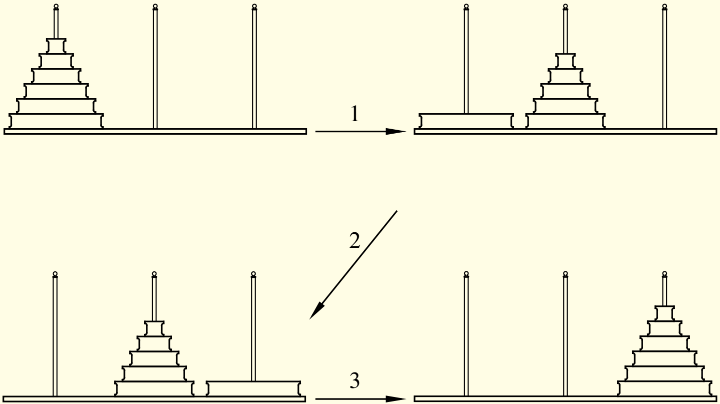

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$$T(1) = 1$$

Solving Recurrences: Plug and Chug

- Expand the recurrence out for a few steps
- Identify the pattern
- Guess a solution based on the pattern
- Check the solution for a few small values of n
- Verify using induction

Plug and Chug for Tower of Hanoi Recurrence

$$T(n) = 2T(n-1) + 1, \qquad T(1) = 1$$

Examples of Recurrences for Runtimes: Exponentiation

$$exp(x, n) = \begin{cases} 1, & \text{if } n = 0 \\ x * exp(x, n-1), & \text{otherwise} \end{cases}$$

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Examples of Recurrences for Runtimes: Fast Exponentiation

$$fexp(x, n) = \begin{cases} 1, & \text{if } n = 0 \\ x, & \text{if } n = 1 \\ fexp(x * x, n/2), & \text{if } n \text{ is even} \\ fexp(x * x, n/2) * x, & \text{if } n \text{ is odd} \end{cases}$$

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Solving Linear Recurrences

- *Homogeneous* linear recurrences are of the form

$$f(n) = \sum_{i=1}^{d} a_i f(n-i)$$

  - Example: Fibonacci series $F(n) = F(n-1) + F(n-2)$

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$$\begin{aligned}
x^n &= \sum_{i=1}^{d} a_i x^{n-i} \\
x^d &= \sum_{i=1}^{d} a_i x^{d-i} \qquad \text{(Dividing all terms by } x^{n-d}\text{)} \\
\sum_{i=0}^{d} a_i x^{d-i} &= 0 \qquad \text{(Rearrange terms to arrive at a polynomial, with } a_0 = 1\text{)}
\end{aligned}$$

Solving Homogeneous Linear Recurrences (Contd.)

- Find the roots $r_1, \ldots, r_d$ of of this polynomial $\sum_{i=0}^{d} a_i x^{d-i} = 0$

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- Note: if the polynomial has fewer than d roots, the general form of the solution gets more complicated — we will ignore this case here.

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5. Thus, the solution is

$$f(n) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Observations about Fibonacci Recurrence Solution

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- So, for larger n, $f(n)$ is determined almost entirely by the first term $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$
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  - $p^n/\sqrt{5}$ is very close to an integer value, although p is irrational!
- The ratio between successive Fibonacci numbers converges to $p = 1.618$, which is called the *golden ratio*

Asymptotic Complexity

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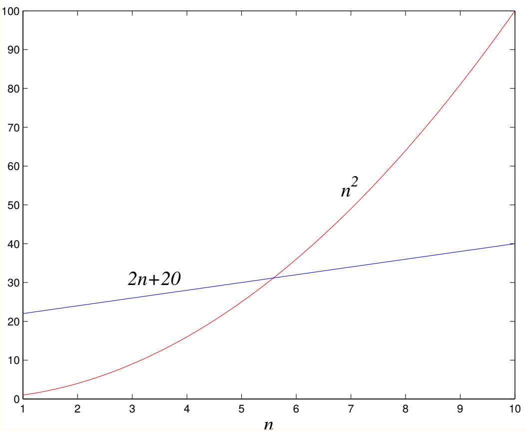
  - But it is too complex to worry about the details, esp. because they differ across programming languages, processor types, etc.
- Why not simplify further?
  - Capture just the growth rate of $T(n)$ as a function of n
  - Ignore constant factors
    - No need to count operations in a loop (their number should be bounded by a constant)
  - Ignore exceptions from the formula for small values of n

Asymptotic Complexity: Big-O notation

Definition

Given functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, *we say* $f = O(g)$, *i.e., "f grows no faster than g,"* *iff*

$$\lim_{x\to\infty} f(x)/g(x) < c \text{ for some constant } c$$



Big-O notation: Examples

- $10n = O(n)$
- $0.0001n^3 + n = O(n^3)$
- $2^n + 10^n + n^2 + 2 = O(10^n)$
- $0.0001n \log n + 10000n = O(n \log n)$

Solving Divide-and-Conquer Recurrences: Master Theorem

If $T(n) = aT\left(\frac{n}{b}\right) + O(n^d)$ for constants $a > 0$, $b > 1$, and $d \geq 0$, then

$$T(n) = \begin{cases} O(n^d), & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

Solving Recurrences: Examples Using Master Theorem

$$T(n) = 2T(n/2) + n$$

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Solving Recurrences: Examples Using Master Theorem

$$T(n) = 4T(n/2) + n^3$$

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Solving Recurrences: Examples Using Master Theorem

$$T(n) = 3T(n/2) + n$$

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