Propositions (Textbook Chapter 1)

A *proposition* is a statement that is either true or false

- **Non-propositions**
  - Sky is beautiful!
  - Tomorrow will be sunny.

- **Examples of propositions**
  - $2 + 3 = 5$
  - $n^2 + n + 41$ is always prime
Claims, Conjectures and Theorems (all propositions)

Conjecture: $a^4 + b^4 + c^4 = d^4$ has no solutions if $a, b, c$ and $d$ are all positive integers [Euler]

---

Claims, Conjectures and Theorems (all propositions)

Conjecture:  \( a^4 + b^4 + c^4 = d^4 \) has no solutions if \( a, b, c \) and \( d \) are all positive integers [Euler]

- Shown false after 200+ years for \( a = 95800, b = 217519, c = 414560 \) and \( d = 422481 \).

---

Claims, Conjectures and Theorems (all propositions)

Conjecture: $a^4 + b^4 + c^4 = d^4$ has no solutions if $a$, $b$, $c$ and $d$ are all positive integers \[\text{[Euler]}\]

- Shown false after 200+ years for $a = 95800$ , $b = 217519$, $c = 414560$ and $d = 422481$.

Four color theorem: Every map can be colored with at most 4 colors while ensuring that no two adjacent regions have the same color.

---


**Claims, Conjectures and Theorems (all propositions)**

**Conjecture:** \( a^4 + b^4 + c^4 = d^4 \) has no solutions if \( a, b, c \) and \( d \) are all positive integers [Euler]

- Shown false after 200+ years for \( a = 95800, b = 217519, c = 414560 \) and \( d = 422481 \).

**Four color theorem:** Every map can be colored with at most 4 colors while ensuring that no two adjacent regions have the same color.

- Shown to be true using software\(^1\).

---

\(^1\)“Four Colors Suffice. How the Map Problem was Solved,” Robin Wilson, Princeton Univ. Press, 2003.

Claims, Conjectures and Theorems (all propositions)

**Conjecture:** \( a^4 + b^4 + c^4 = d^4 \) has no solutions if \( a, b, c \) and \( d \) are all positive integers [Euler]

- Shown false after 200+ years for \( a = 95800, b = 217519, c = 414560 \) and \( d = 422481 \).

**Four color theorem:** Every map can be colored with at most 4 colors while ensuring that no two adjacent regions have the same color.

- Shown to be true using software\(^1\).

**Fermat’s Theorem:** \( x^n + y^n = z^n \) has no integral solutions for \( n > 2 \).

\(^1\)“Four Colors Suffice. How the Map Problem was Solved,” Robin Wilson, Princeton Univ. Press, 2003.

Conjecture: \( a^4 + b^4 + c^4 = d^4 \) has no solutions if \( a, b, c \) and \( d \) are all positive integers [Euler]

- Shown false after 200+ years for \( a = 95800 \), \( b = 217519 \), \( c = 414560 \) and \( d = 422481 \).

Four color theorem: Every map can be colored with at most 4 colors while ensuring that no two adjacent regions have the same color.

- Shown to be true using software\(^1\).

Fermat’s Theorem: \( x^n + y^n = z^n \) has no integral solutions for \( n > 2 \).

- Fermat omitted the proof in 1630 because “it did not fit in the margin”
- Remained unproven for 300+ years\(^2\).

\(^1\)“Four Colors Suffice. How the Map Problem was Solved,” Robin Wilson, Princeton Univ. Press, 2003.
Claims, Conjectures and Theorems (all propositions)

Conjecture: \( a^4 + b^4 + c^4 = d^4 \) has no solutions if \( a, b, c \) and \( d \) are all positive integers [Euler]
- Shown false after 200+ years for \( a = 95800, b = 217519, c = 414560 \) and \( d = 422481 \).

Four color theorem: Every map can be colored with at most 4 colors while ensuring that no two adjacent regions have the same color.
- Shown to be true using software\(^1\).

Fermat’s Theorem: \( x^n + y^n = z^n \) has no integral solutions for \( n > 2 \).
- Fermat omitted the proof in 1630 because “it did not fit in the margin”
- Remained unproven for 300+ years\(^2\).

Goldbach’s Conjecture: Every even integer greater than 2 is the sum of two primes.

\(^1\)“Four Colors Suffice. How the Map Problem was Solved,” Robin Wilson, Princeton Univ. Press, 2003.
Claims, Conjectures and Theorems (all propositions)

Conjecture: $a^4 + b^4 + c^4 = d^4$ has no solutions if $a, b, c$ and $d$ are all positive integers [Euler]
- Shown false after 200+ years for $a = 95800$, $b = 217519$, $c = 414560$ and $d = 422481$.

Four color theorem: Every map can be colored with at most 4 colors while ensuring that no two adjacent regions have the same color.
- Shown to be true using software\(^1\).

Fermat’s Theorem: $x^n + y^n = z^n$ has no integral solutions for $n > 2$.
- Fermat omitted the proof in 1630 because “it did not fit in the margin”
- Remained unproven for 300+ years\(^2\).

Goldbach’s Conjecture: Every even integer greater than 2 is the sum of two primes.
- Holds for numbers up to $10^{18}$, but unknown if it is always true

---

\(^1\)“Four Colors Suffice. How the Map Problem was Solved,” Robin Wilson, Princeton Univ. Press, 2003.
Logical Formulas (Textbook Chapter 3)

- Obtained by combining propositions using logical connectives (aka logical operators)
  - ∧ ("and" operation)
  - ∨ ("or" operation)
  - ¬ ("not" operation)
  - → ("implies" operation)
English to Logic Formulas

- If humans are mortal **and** Greeks are human **then** Greeks are mortal
Conditional statement \((P \rightarrow Q)\)

- \(P\) is the **hypothesis/premise/antecedent**, \(Q\) is the **conclusion/consequence**
- \(P \rightarrow Q\) is also called:

<table>
<thead>
<tr>
<th>“if (P), then (Q)”</th>
<th>“(P) implies (Q)”</th>
</tr>
</thead>
<tbody>
<tr>
<td>“(P) only if (Q)”</td>
<td>“(if (P), (Q))”</td>
</tr>
<tr>
<td>“(Q) follows from (P)”</td>
<td>“(Q), provided that (P)”</td>
</tr>
<tr>
<td>“(not (P) unless (Q))”</td>
<td>“(Q) if/when/whenever (Q)”</td>
</tr>
<tr>
<td>“(P) is sufficient for (Q)”</td>
<td>“a sufficient condition for (Q) is (P)”</td>
</tr>
<tr>
<td>“(Q) is necessary for (P)”</td>
<td>“a necessary condition for (P) is (Q)”</td>
</tr>
</tbody>
</table>
Understanding Conditionals

- What is the intuitive meaning of $P \rightarrow Q$?
  - Conditional statement is like a promise
  - Under what circumstances is the promise kept/broken?
  - Example: “If tomorrow is sunny, I will take you to the beach.”

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \rightarrow Q$</th>
</tr>
</thead>
</table>
| Tomorrow is sunny | Go to the beach         | Promise is kept (T)
| Tomorrow is sunny | Did not go to the beach | Promise is broken (F)
| Tomorrow is not sunny | Go to the beach     | Promise is not broken (T)
| Tomorrow is not sunny | Did not go to the beach | Promise is not broken (T)

- $P \rightarrow Q$ being true because $P$ is false is called vacuously true or true by default
Contrapositive, Inverse and Converse

Definitions

- **Contrapositive** of $P \rightarrow Q$ is $\neg q \rightarrow \neg p$
- **Converse** of $P \rightarrow Q$ is $q \rightarrow p$
- **Inverse** of $P \rightarrow Q$ is $\neg p \rightarrow \neg q$

Identities

- Conditional $\equiv$ Contrapositive $\quad \triangleright$ Useful for proofs
- Conditional $\not\equiv$ Converse
- Conditional $\not\equiv$ Inverse
- Converse $\equiv$ Inverse
Examples of Contrapositive, Inverse and Converse

- **Conditional ≡ Contrapositive.**
  
  “If tomorrow is sunny, we will go to the beach.”
  “If we don’t go to the beach tomorrow, then it is not sunny.”

- **Converse ≡ Inverse.**
  
  “If we go to the beach tomorrow, then it is sunny.”
  “If tomorrow is not sunny, then we will not go to the beach.”

- **Conditional ≡ Contrapositive.**
  
  “If $x > 2$, then $x^2 > 4$. ” ▷ True
  “If $x^2 \leq 4$, then $x \leq 2$. ” ▷ True

- **Converse ≡ Inverse.**
  
  “If $x^2 > 4$, then $x > 2$. ” ▷ False
  “If $x \leq 2$, then $x^2 \leq 4$. ” ▷ False
Necessary and Sufficient Conditions

- **P is a sufficient condition for Q** means $P \rightarrow Q$
- **P is a necessary condition for Q** means $\neg P \rightarrow \neg Q$
- **P only if Q** means $P \rightarrow Q$
  - Equivalently, if $P$ then $Q$
- For real $x$, $x = 1$ is a sufficient condition for $x^2 = 1$
  - i.e., If $x = 1$ then $x^2 = 1$ $\triangleright$ True
- For real $x$, $x^2 = 1$ is a necessary condition for $x = 1$
  - i.e., If $x^2 \neq 1$ then $x \neq 1$ $\triangleright$ True
- For real $x$, $x = 1$ only if $x^2 = 1$
  - i.e., If $x^2 \neq 1$, then $x \neq 1$ $\triangleright$ True
English to Logic Formulas

\( P \) ::= “you get an A in the final exam”

\( Q \) ::= “you do every problem in the book”

\( R \) ::= “you get an A in the course”

- If you do every problem in the book, you will get an A in the final exam

- You got an A in the course but you did not do every problem in the book

- To get an A in the class, it is necessary to get an A on the final.
Modeling Problems in Propositional Logic

You can’t locate your glasses. You know the following statements are true:

(a) If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

(b) If my glasses are on the kitchen table, then I saw them at breakfast.

(c) I did not see my glasses at breakfast.

(d) I was reading the newspaper in the living room or the kitchen.

(e) If I was reading the newspaper in the living room then my glasses are on the coffee table.

Where are the glasses?
Modeling Problems in Propositional Logic

Let:

- RK = I was reading the newspaper in the kitchen.
- GK = My glasses are on the kitchen table.
- SB = I saw my glasses at breakfast.
- RL = I was reading the newspaper in the living room.
- GC = My glasses are on the coffee table.
Modeling Problems in Propositional Logic

(a) If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table: $RK \rightarrow GK$

(b) If my glasses are on the kitchen table, then I saw them at breakfast: $GK \rightarrow SB$

(c) I did not see my glasses at breakfast: $\neg SB$

(d) I was reading the newspaper in the living room or the kitchen: $RL \lor RK$

(e) If I was reading the newspaper in the living room then my glasses are on the coffee table: $RL \rightarrow GC$
Modeling Problems in Propositional Logic

(a) If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table: \( RK \rightarrow GK \)

(b) If my glasses are on the kitchen table, then I saw them at breakfast: \( GK \rightarrow SB \)

(c) I did not see my glasses at breakfast: \( \neg SB \)

(d) I was reading the newspaper in the living room or the kitchen: \( RL \lor RK \)

(e) If I was reading the newspaper in the living room then my glasses are on the coffee table: \( RL \rightarrow GC \)

\( \neg SB \)
Modeling Problems in Propositional Logic

(a) If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table: \( RK \rightarrow GK \)

(b) If my glasses are on the kitchen table, then I saw them at breakfast: \( GK \rightarrow SB \)

(c) I did not see my glasses at breakfast: \( \neg SB \)

(d) I was reading the newspaper in the living room or the kitchen: \( RL \lor RK \)

(e) If I was reading the newspaper in the living room then my glasses are on the coffee table: \( RL \rightarrow GC \)

\( \neg SB \)

\( \neg SB \rightarrow \neg GK \)
Modeling Problems in Propositional Logic

(a) If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table: $\text{RK} \rightarrow \text{GK}$

(b) If my glasses are on the kitchen table, then I saw them at breakfast: $\text{GK} \rightarrow \text{SB}$

(c) I did not see my glasses at breakfast: $\neg \text{SB}$

(d) I was reading the newspaper in the living room or the kitchen: $\text{RL} \lor \text{RK}$

(e) If I was reading the newspaper in the living room then my glasses are on the coffee table: $\text{RL} \rightarrow \text{GC}$

- $\neg \text{SB}$

- From $\text{GK} \rightarrow \text{SB}$, conclude $\neg \text{SB} \rightarrow \neg \text{GK}$

- From the above two, conclude $\neg \text{GK}$
Modeling Problems in Propositional Logic

(a) If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table: $\text{RK} \rightarrow \text{GK}$

(b) If my glasses are on the kitchen table, then I saw them at breakfast: $\text{GK} \rightarrow \text{SB}$

(c) I did not see my glasses at breakfast: $\neg \text{SB}$

(d) I was reading the newspaper in the living room or the kitchen: $\text{RL} \lor \text{RK}$

(e) If I was reading the newspaper in the living room then my glasses are on the coffee table: $\text{RL} \rightarrow \text{GC}$

- $\neg \text{SB}$
- From $\text{GK} \rightarrow \text{SB}$, conclude $\neg \text{SB} \rightarrow \neg \text{GK}$
- From the above two, conclude $\neg \text{GK}$
- Use (a) in a similar manner: from $\neg \text{GK}$ and $\text{RG} \rightarrow \text{GK}$, conclude $\neg \text{RK}$. 
Modeling Problems in Propositional Logic

(a) If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table: \( RK \rightarrow GK \)

(b) If my glasses are on the kitchen table, then I saw them at breakfast: \( GK \rightarrow SB \)

(c) I did not see my glasses at breakfast: \( \neg SB \)

(d) I was reading the newspaper in the living room or the kitchen: \( RL \lor RK \)

(e) If I was reading the newspaper in the living room then my glasses are on the coffee table: \( RL \rightarrow GC \)

- \( \neg SB \)
- From \( GK \rightarrow SB \), conclude \( \neg SB \rightarrow \neg GK \)
- From the above two, conclude \( \neg GK \)
- Use (a) in a similar manner: from \( \neg GK \) and \( RG \rightarrow GK \), conclude \( \neg RK \).
- From \( RL \lor RK \) and \( \neg RK \), conclude \( RL \).
Modeling Problems in Propositional Logic

(a) If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table: \( RK \rightarrow GK \)

(b) If my glasses are on the kitchen table, then I saw them at breakfast: \( GK \rightarrow SB \)

(c) I did not see my glasses at breakfast: \( \neg SB \)

(d) I was reading the newspaper in the living room or the kitchen: \( RL \lor RK \)

(e) If I was reading the newspaper in the living room then my glasses are on the coffee table: \( RL \rightarrow GC \)

\( \neg SB \)

From \( GK \rightarrow SB \), conclude \( \neg SB \rightarrow \neg GK \)

From the above two, conclude \( \neg GK \)

Use (a) in a similar manner: from \( \neg GK \) and \( RG \rightarrow GK \), conclude \( \neg RK \).

From \( RL \lor RK \) and \( \neg RK \), conclude \( RL \).

From \( RL \) and (e), conclude \( GC \). So, look on the coffee table!
Example: Truth tellers and liars

There is an island that consists of liars and truth tellers:
- Liars always lie.
- Truth who always tell the truth

You visit the island and are approached by two natives A and B:
- A says: B is a truth teller.
- B says: A and I are of opposite types.

What are A and B?
Suppose $A$ is a truth teller.

- What $A$ says is true. $\triangleright$ by definition of truth teller
- So $B$ is also a truth teller. $\triangleright$ That’s what $A$ said.
- So, what $B$ says is true. $\triangleright$ by definition of truth teller
- So, $A$ and $B$ are of opposite types. $\triangleright$ That’s what $B$ said.
- **Contradiction:** $A$ and $B$ are both truth tellers and $A$ and $B$ are of opposite type.

So, initial assumption is false. $\triangleright$ by the contradiction rule

- So $A$ is not a truth teller. $\triangleright$ negation of assumption
- So $A$ is a liar. $\triangleright$ by elimination: All inhabitants are truth tellers or liars, so since $A$ is not a truth teller, $A$ is a liar.
- So What $A$ says is false.
- So $B$ is not a truth teller.
- So $B$ is also a liar. $\triangleright$ by elimination

**Final answer:** $A$ and $B$ are both liars
## Truth Tables

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \rightarrow Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$\neg P$</th>
<th>$\neg P \lor Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The table on the left shows the truth values for the implication $P \rightarrow Q$. The table on the right shows the truth values for the disjunction $\neg P \lor Q$. 

These tables illustrate the use of truth tables in representing logical expressions and their truth values.
Using Truth Tables to Evaluate Logical Formulas

Does $P \rightarrow Q$ imply $\neg Q \rightarrow \neg P$?

All the two formulas equivalent?
Using Truth Tables to Evaluate Logical Formulas

Does $P \rightarrow Q$ imply $\neg P \rightarrow \neg Q$?
Using Truth Tables to Show Equivalence

What about \( \neg(P \land Q) \) and \( \neg P \lor \neg Q \)?

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( \neg P )</th>
<th>( \neg Q )</th>
<th>( \neg(P \land Q) )</th>
<th>( \neg P \lor \neg Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

The truth tables for \( \neg(P \land Q) \) and \( \neg P \lor \neg Q \) match, so we conclude they are equivalent:

\[ \neg(P \land Q) \iff \neg P \lor \neg Q \]  

[De Morgan’s Law]
De Morgan’s Law Examples for Practice

- $\neg (P \lor Q)$
- $\neg (P \land Q \land R)$
- $\neg (P \land (Q \rightarrow R))$
Properties of Boolean Operators

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression 1</th>
<th>Expression 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Commutativity</strong></td>
<td>$P \lor Q \leftrightarrow Q \lor P$</td>
<td>$P \land Q \leftrightarrow Q \land P$</td>
</tr>
<tr>
<td><strong>Associativity</strong></td>
<td>$P \lor (Q \lor R) \leftrightarrow (P \lor Q) \lor R$</td>
<td>$P \land (Q \land R) \leftrightarrow (P \land Q) \land R$</td>
</tr>
<tr>
<td><strong>Distributivity</strong></td>
<td>$P \lor (Q \land R) \leftrightarrow (P \lor Q) \land (P \lor R)$</td>
<td>$P \land (Q \lor R) \leftrightarrow (P \land Q) \lor (P \land R)$</td>
</tr>
<tr>
<td><strong>De Morgan’s Laws</strong></td>
<td>$\neg (P \lor Q) \leftrightarrow \neg P \land \neg Q$</td>
<td>$\neg (P \land Q) \leftrightarrow \neg P \lor \neg Q$</td>
</tr>
</tbody>
</table>

- Compare these laws with those for arithmetic, with ‘+’ for ‘\lor’ and ‘*’ for ‘\land’.

- Which of the properties hold? Which ones don’t?
Additional Useful Identities

\[
\neg\neg P \iff P \\
P \lor \neg P \iff \text{true} \\
P \land \neg P \iff \text{false} \\
P \lor P \iff P \\
P \land P \iff P \\
\text{true} \lor P \iff \text{true} \\
\text{false} \lor P \iff P \\
\text{true} \land P \iff P \\
\text{false} \land P \iff \text{false} \\
P \rightarrow Q \iff \neg P \lor Q \\
\text{true} \rightarrow P \iff P \\
\text{false} \rightarrow P \iff \text{true} \\
P \rightarrow \text{true} \iff \text{true}
\]
Disjunctive Normal Form (DNF)

- Formulas of the form
  \[ \psi_1 \lor \psi_2 \lor \cdots \lor \psi_n \]
  where each \( \psi \) is a conjunction of (possibly negated) propositions.

- Example: \( P_1 \land \neg P_2 \land P_3 \lor \neg P_1 \lor P_3 \)

- The only operator permitted at the top level is disjunction

- Only the conjunction operator is permitted at the next level
  - Only propositional variables or their negations at the third level

- Any propositional formula can be transformed into an equivalent formula in DNF.
  - Conversion repeatedly uses the identities from previous slides.
Conjunctive Normal Form (CNF) and the SAT problem

- Formulas are of the form
  \[ \psi_1 \land \psi_2 \land \cdots \land \psi_n \]
  where each \( \psi \) is a conjunction of (possibly negated) propositions.
- Example: \( P_1 \land \neg P_2 \land P_3 \)
- Any propositional formula can be transformed into an equivalent formula in CNF.
  - Use boolean operator properties systematically.
- SAT problem: Given a CNF formula, determine if it is satisfiable.
  - No efficient algorithm known
  - Forms the basis of NP-completeness and the \( P \neq NP \) hypothesis
Validity, Satisfiability and Equivalence

- A formula $\varphi$ is **valid** iff it is true for all possible values of propositions in them
  - Example: $P \lor \neg P$

- A formula $\varphi$ is **satisfiable** iff it is true for some values of the propositions in them
  - Most formulas are satisfiable
  - Example: $P \rightarrow Q$

- A formula $\varphi$ is **equivalent** to $\psi$ iff they have the exact same value for all possible values of the propositions contained in them
  - In other words, the truth tables for $\varphi$ and $\psi$ match fully
  - We saw several examples in the previous slides
Axiom: a proposition accepted to be true.

- Usually, no way to prove them; and they seem obviously true.
- Example: there exists a straight line between any two points
Axioms, Inference Rules, Theorems and Proofs (Textbook §1.3)

Axiom: a proposition accepted to be true.

- Usually, no way to prove them; and they seem obviously true.
- Example: there exists a straight line between any two points

Inference rule: an axiom to derive new propositions from existing ones

\[ \vdash P, \vdash P \rightarrow Q \quad \Rightarrow \quad \vdash Q \quad (modus ponens) \]
Axioms, Inference Rules, Theorems and Proofs (Textbook §1.3)

Axiom: a proposition accepted to be true.
- Usually, no way to prove them; and they seem obviously true.
- Example: there exists a straight line between any two points

Inference rule: an axiom to derive new propositions from existing ones
\[
\frac{\vdash P, \vdash P \rightarrow Q}{\vdash Q} \quad (\text{modus ponens})
\]

Theorems, Lemmas: Propositions that can be derived from axioms using inference rules
Axioms, Inference Rules, Theorems and Proofs (Textbook §1.3)

**Axiom:** a proposition accepted to be true.
- Usually, no way to prove them; and they seem obviously true.
- Example: there exists a straight line between any two points

**Inference rule:** an axiom to derive new propositions from existing ones

\[
\vdash P, \vdash P \rightarrow Q \\
\therefore \vdash Q 
\]

(modus ponens)

**Theorems, Lemmas:** Propositions that can be derived from axioms using inference rules

**(Formal) Proof:** The exact manner in which a theorem was derived from axioms.
What is a valid argument?

**Definition**

- An argument is **valid** if the conclusion follows necessarily from the premises.
Valid argument: Examples

- If Socrates is a man, then Socrates is mortal.
  Socrates is a man.
  Therefore, Socrates is mortal.  ▶ Valid argument

- If Socrates is a man, then Socrates is mortal.
  Socrates is mortal.
  Therefore, Socrates is a man.  ▶ Invalid argument

- If Socrates is a man, then Socrates is mortal.
  Socrates is not mortal.
  Therefore, Socrates is not a man.  ▶ Valid argument

- If Socrates is a man, then Socrates is mortal.
  Socrates is not a man.
  Therefore, Socrates is not mortal.  ▶ Invalid argument
Valid argument: Examples

- If it is raining, then it is cloudy.
  It is raining.
  Therefore, it is cloudy. ▷ Valid argument

- If it is raining, then it is cloudy.
  It is cloudy.
  Therefore, it is raining. ▷ Invalid argument

- If it is raining, then it is cloudy.
  It is not cloudy.
  Therefore, it is not raining. ▷ Valid argument

- If it is raining, then it is cloudy.
  It is not raining.
  Therefore, it is not cloudy. ▷ Invalid argument
Valid argument: Examples

- If $x > 2$, then $x^2 > 4$.
  
  $x > 2$.
  
  Therefore, $x^2 > 4$. ▷ Valid argument

- If $x > 2$, then $x^2 > 4$.
  
  $x^2 > 4$.
  
  Therefore, $x > 2$. ▷ Invalid argument

- If $x > 2$, then $x^2 > 4$.
  
  $x^2 \leq 4$.
  
  Therefore, $x \leq 2$. ▷ Valid argument

- If $x > 2$, then $x^2 > 4$.
  
  $x \leq 2$.
  
  Therefore, $x^2 \leq 4$. ▷ Invalid argument
Valid argument: Examples

• If $P$, then $Q$.
  
  $P$.
  
  Therefore, $Q$. ▶ Valid argument

• If $P$, then $Q$.
  
  $Q$.
  
  Therefore, $P$. ▶ Invalid argument

• If $P$, then $Q$.
  
  $\neg Q$.
  
  Therefore, $\neg P$. ▶ Valid argument

• If $P$, then $Q$.
  
  $\neg P$.
  
  Therefore, $\neg Q$. ▶ Invalid argument
Proving an Implication $P \rightarrow Q$

- Strategy 1: Assume $P$, show that $Q$ follows
- Example: If $2 < x < 4$ then $x^2 - 6x + 8 < 0$
Proving an Implication $P \rightarrow Q$

- **Strategy 2**: Prove the contrapositive $\neg Q \rightarrow \neg P$

- **Example**: If $r$ is irrational then $\sqrt{r}$ is irrational
Proving $P$ iff $Q$ ("$P$ if and only if $Q$"")

- $P \iff Q$ is proved by showing $P \implies Q$ and then $Q \implies P$

- Example: $2 < x < 4$ iff $x^2 - 6x + 8 < 0$
Proof by Cases

- To prove $P \rightarrow Q$ when $P$ is complex

- We can simplify the proof by “breaking up” $P$ into cases:
  - Find $P_1, P_2$ such that $P \rightarrow P_1 \lor P_2$
  - Prove $P_1 \rightarrow Q$ and $P_2 \rightarrow Q$
  - Note $P_1$ and $P_2$ can overlap, i.e., they can simultaneously be true.
    - But most proofs consider mutually exclusive cases
  - $P_i$’s must be exhaustive, i.e., cover every possible case when $P$ could be true
Proof by Cases

Example: \( \max(r, s) + \min(r, s) = r + s \)
False Hypothesis and Vacuous Truth

What happens to $P \rightarrow Q$ when $P$ is false?

- In this case, $P \rightarrow Q$ holds *vacuously*

- So, $F \rightarrow Q$ for any $Q$!

- If $P$ is false, then $P \rightarrow \neg P$ holds!

  - Take the contrapositive of this, you get

- Basis of proof-by-contradiction strategy
Proof by Contradiction

Example: Show that there are infinitely many primes
Idea: Circuits and logic are related

Open or off or false

Closed or on or true
Idea: Circuits and logic are related

<table>
<thead>
<tr>
<th>Switches</th>
<th>Light bulb</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>$Q$</td>
</tr>
<tr>
<td>closed</td>
<td>closed</td>
</tr>
<tr>
<td>closed</td>
<td>open</td>
</tr>
<tr>
<td>open</td>
<td>closed</td>
</tr>
<tr>
<td>open</td>
<td>open</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Switches</th>
<th>Light bulb</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>$Q$</td>
</tr>
<tr>
<td>closed</td>
<td>closed</td>
</tr>
<tr>
<td>closed</td>
<td>open</td>
</tr>
<tr>
<td>open</td>
<td>closed</td>
</tr>
<tr>
<td>open</td>
<td>open</td>
</tr>
</tbody>
</table>
Birth of digital logic circuits

- **1930s**: Mechanical switches were used in circuit design

- **Late 1930s**: Great idea that mathematical logic (or Boolean algebra) can be used to analyze switches

- **1940s and 1950s**: Electronic switches for circuit design
  - Led to the development of electronic computers, electronic telephone switching systems, traffic light controls, electronic calculators, and the control mechanisms

- Electronic switches to implement logic is the fundamental concept that underlies all electronic digital computers
Evolution of electronic computers

- Vacuum tube switches (1940s on)
- Semiconductor switches (transistors) from 1950s ...
- Integrated circuits from 1960s
- The number of transistors have increased by 2x every two years
  - Predicted by Gordon Moore (Moore’s Law) (1965)
  - Intel 4004 processor had 2250 gates in 1971, about 10\(\mu\)m
  - Today’s microprocessors have more than 100 billion transistors, about 10nm!
  - Solid state drives have over 2 trillion transistors
Complicated logic gates as black boxes

A black box focuses on the **functionality** and ignores the **hardware implementation details**
Simple logic gates

Method

- Complicated logic gates can be built using a collection of simple logic gates such as NOT-gate, AND-gate, and OR-gate

\[
P \rightarrow \neg P
\]

\[
P \land Q
\]

\[
P \lor Q
\]
Combinational Vs Sequential Logic

- **Combinational circuit**: output is purely a function of current inputs
  - Combines inputs using a series of gates
  - No output of a gate can eventually feed back into that gate.

- **Sequential circuits**: output feeds back into input, so it depends on current *and* previous inputs.
  - Basis of memory and sequential instruction processing
    - Basic unit is called a flip-flop, which in turn is realized using gates
  - Divides computation into steps
  - Progress from one step to next is governed by a clock
Problem-solving in digital logic circuits

Diagram $\rightarrow$ Expression $\rightarrow$ Table

Physical circuit design $\rightarrow$ Electronic functionality
Problem-solving in digital logic circuits

- **Circuit → Table**
  - Logic circuit → Boolean expression
  - Simplify Boolean expression
  - Boolean expression → Input-output table

- **Table → Circuit**
  - Input-output table → Boolean expression
  - Simplify Boolean expression
  - Boolean expression → Logic circuit
Problem

Determine the input-output table for the given logic circuit.
Circuit $\rightarrow$ Table

- Circuit $\rightarrow$ expression

```
P Q    P ∨ Q    P ∧ Q    ¬(P ∧ Q)  (P ∨ Q) ∧ ¬(P ∧ Q)
1 1    1       1       0          0
1 0    1       0       1          1
0 1    1       0       1          1
0 0    0       0       1          0
```

- Simplify expression: \((P ∨ Q) ∧ ¬(P ∧ Q) \equiv P \oplus Q\) $\Rightarrow$ Exclusive or
### Problem

Determine the logic circuit for the given input-output table.

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>Q</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
**Table → Circuit**

1. **Table → expression**

\[(P \land Q \land R) \lor (P \land \neg Q \land R) \lor (P \land \neg Q \land \neg R)\]

Disjunctive normal form or sum-of-products form

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P</td>
<td>Q</td>
<td>R</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
2. Expression → circuit
2. Simplify expression

\[(P \land Q \land R) \lor (P \land \neg Q \land R) \lor (P \land \neg Q \land \neg R)\]

\[\equiv P \land (\neg Q \lor R) \quad \triangleright \text{How?}\]

3. Expression → circuit

![Circuit Diagram]

\[P, Q, R\]
Equivalence of logic circuits

- Two digital logic circuits are called equivalent if and only if their input-output tables are identical.
- We can use boolean simplification as well!
- Show that the following two logic circuits are equivalent.
Equivalence of logic circuits

- Write this 8-input AND gate using 2-input AND gates only.
NAND and NOR gates

- **NAND**: \( \overline{P \land Q} \)
- **NOR**: \( \overline{P \lor Q} \)

**Note**: Every boolean function can be realized entirely using NAND gates.

- Same holds for NOR as well.

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>( Q )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>( Q )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Logic and programming

- Is there way to simplify

  \[
  \text{if (!((x \geq 0) && (x \leq 10)) || (x \geq 20))}
  \]

- What about

  \[
  \begin{align*}
  \text{if !((x \leq 20) || ((x \geq 30) && (x \leq 39)))} \\
  \text{if ((x \geq 20) && (x \leq 30)) || (x \geq 40))}
  \end{align*}
  \]
Can the following circuit be optimized?
Logic and Reasoning

If it is raining, then it is cloudy.  
It is not cloudy.  
So, it is not raining.

Steve  
Natasha  
John

Is John’s conclusion logical?
Logic and proofs

- Proving implications: \[ \frac{P \vdash Q}{\vdash P \rightarrow Q} \]

- Proving implication by showing the contrapositive: \[ \frac{\neg Q \vdash \neg P}{\vdash P \rightarrow Q} \]

- Case-splitting: \[ \frac{P \land Q \vdash R, \ P \land \neg Q \vdash R}{\vdash P \rightarrow R} \]

- Establishing equivalence: \[ \frac{\vdash P \rightarrow Q, \ \vdash Q \rightarrow P}{\vdash P \leftrightarrow Q} \]

- Proof by contradiction: \[ \frac{P \vdash \neg P}{\vdash \neg P} \]
Unit Summary

- Propositions, claims, conjectures and theorems
- Logical formulas
  - English to logical formulas
  - Truth tables: construction and use
  - Validity, satisfiability and equivalence
  - Equivalences among logical operators
    - DNF, CNF and SAT
- Axioms, inference rules and proofs
- Proof techniques
- Digital circuits