Predicates (Textbook §3.6)

A *predicate* is a *proposition* whose truth depends on the values of variables

P

$$P(n) ::= \text{“}n \text{ is a perfect square”}$$

P(n) holds for n = 0, 1, 4, 9, 16, ...

1 is a perfect square ← true prop.

3 is a perfect square ← false prop

Predicates (Textbook §3.6)

A *predicate* is a *proposition* whose truth depends on the values of variables

$$P(n) ::= \text{“}n \text{ is a perfect square”}$$

one

- If P is a unary predicate, $P(n)$, being a proposition, is either true or false.

Predicates (Textbook §3.6)

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$Py(3,4,5)$ T

$Py(2,3,4)$ F

$$P(n) ::= \text{“}n \text{ is a perfect square”}$$

- If P is a unary predicate, $P(n)$, being a proposition, is either true or false.
- A k-ary predicate takes k variables, e.g.,

$$Py(x, y, z) ::= \text{“}x^2 + y^2 = z^2 \text{ for integers } x, y \text{ and } z\text{”}$$

is a ternary predicate that characterizes Pythagorean triples. x, y and z are said to be *free variables* in the formula defining Py.

$$Q(x,y) ::= x \geq y \vee y > x$$

Satisfiability and Validity

These terms have the same high level meaning as in the propositional case.

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- Related to *existential* quantification
- Example: Are there values of x, y and z such that $Py(x, y, z)$? $R(x) ::= x > x$

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- Related to *existential* quantification
- Example: Are there values of x, y and z such that $Py(x, y, z)$?

Valid: The formula is true for all possible values of variables

- Related to *universal* quantification.

$$\text{Example:} Q(x, y) ::= \text{“if } y > 0 \text{ then } x + y > x\text{”}$$

Is this always true?

Quantifiers

$\forall x \in \mathbb{N} \ \forall y \in \mathbb{N} \quad x \geq y \lor y > x$

$x = \sqrt{\frac{7}{5}}$

Universal Quantifier: $\forall$ "for all"

$$\forall x \in \mathbb{R} \ \ x^2 \geq 0$$

Existential Quantifier: $\exists$ "there exists"

Valid $\rightarrow$ $\exists x \in \mathbb{R} \ 5x^2 - 7 = 0$ $\qquad \exists x \in \mathbb{N} \ 5x^2 - 7 = 0$

Unsat

The sets may not be explicitly specified if they are obvious from context.

If multiple variables belong to the same domain, we may abbreviate:

$\forall x, y \in \mathbb{N} \quad x \geq y \lor y > x$

Free Vs Bound Variables

- A quantifier "captures" or "binds" a variable. It is no longer free to take any possible value

$$P(n) ::= \exists x \in \mathbb{N},\ n = x^2$$

  - This formula has two variables n and x, but x has been bound by the quantifier.
  - n continues to be *free:* It can take any value.
    - Some values of n make $P(n)$ true, while others render it false.

Examples of Quantified Statements

“Every American has a dream.”

set of dreams: D

set of Americans: A

$\forall x \in A \ \exists d \in D$
x dreams of d

$\exists d \in D \ \forall x \in A$
x dreams of d

Examples of Quantified Statements

$\boxed{\text{Every even integer greater than 2 is the sum of two primes}}$

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Every even integer greater than 2 is the sum of two primes

- For every even integer n greater than 2, there exist primes p and q s.t. $n = p + q$

$\forall n \in \mathbb{N}$ $n > 2$ and even(n) $\to$ $\exists p, q \in \mathbb{N}$ s.t. p and q are prime and $n = p + q$

$\exists p, q$ $\forall n$ $(n > 2$ and n is even$)$ $\to$ p and q are prime and $n = p + q$

Order of Quantifiers

- Sometimes the order does not matter
  - e.g., when two universal quantifiers are nested

$\forall y \; \forall x \quad x > y \lor y \geq x$

- But other times, they mean a lot! One order will make sense but the other will not.
  - Often, you cannot change the order of existential operators

Negating Quantifiers

set P of persons
pred. $L(x)$ to say "x likes ice cream"

- Not everyone likes ice cream.
- There is someone who does not like ice cream

$\exists n \in P \ \neg L(n)$

$\neg \forall n \in P \ L(n)$

- There is no one who likes being mocked

M

$\neg \exists n \in P \ M(n)$

- Everyone dislikes being mocked

$\forall n \in P \ \neg M(n)$

$\exists$

Negating Quantifiers

- Not everyone likes ice cream.

$$\neg(\boxed{\forall x \in P\ L(x)})$$

$$\cancel{\neg}(\cancel{\neg}\exists x \in P\ \neg L(x))$$

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$$\cancel{\neg}(\cancel{\neg}\forall x \in P\ \neg M(x))$$

$$\forall x \in P\ \neg M(x)$$

- $\forall$ is equivalent to $\neg\exists\neg$
- $\exists$ is equivalent to $\neg\forall\neg$

Unit Summary

- Predicate Vs Proposition
- Satisfiability and Validity
- Quantifiers
  - Free and Bound variables
  - Conversion between English and Logical Formulas
- Order of quantifiers
- Negating quantifiers