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  - $\neg\forall x\ P(x)$ is just an existential formula, so needs just an example
    - An universally quantified formula can be *disproved* with a single (counter) example
  - $\forall x\ P(x)$ *cannot* be proved with an example, as P should hold for *all* values of x.
    - Typically, x ranges over an infinite set, so we cannot explicitly try out all possible x
    - So, we need some insight to develop a logical argument that P holds regardless of the value of x

Proof by Cases

- To prove $P \rightarrow Q$ when P is complex
- We can simplify the proof by "breaking up" P into cases:
  - Find P_1, P_2 such that $P \rightarrow P_1 \vee P_2$
  - Prove $P_1 \rightarrow Q$ and $P_2 \rightarrow Q$
  - Note P_1 and P_2 can overlap, i.e., they can simultaneously be true.
    - But most proofs consider mutually exclusive cases
  - P_i's must be exhaustive, i.e., cover every possible case when P could be true
    - Otherwise $P \rightarrow P_1 \vee P_2$ won't hold.

Proof by Cases

Example: $max(r, s) + min(r, s) = r + s$

Proof by Cases

- If every pair of people in a group has met before, let us call the group a club.
- If no pair has met, let us call it a group of strangers.

Theorem. Every collection of 6 people includes a club of 3 people or a group of 3 strangers.

Proof by Contradiction

For positive numbers a and b, let $n = ab$. Either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

Template for Proofs By Contradiction

1. Start by assuming that the theorem is not true.
   - Your proof should start with "Proof is by contradiction. Assume P is false." where P is the theorem you are trying to prove.
2. Establish a contradiction
   - Show that the negation of the theorem contradicts something that you have assumed or known to be true.
     - Well known identities or laws
     - One of the antecedents of the theorem
     - Negation of the consequent of the theorem
     - $\dots$
3. This contradiction shows that the assumption ($\neg P$) must be false, thus proving P.
4. End your proof with $\square$ or $\blacksquare$ or "Thus proved."

Proof by Contradiction

There are infinitely many primes.

Give an example of two distinct positive integers m and n such that:

- $m < n$
- n^2 is a multiple of m, but n is not a multiple of m.

Proof by Contradiction

$$\forall a, n \in \mathbb{N}\ even(a^n) \Rightarrow even(a)$$

Proof by Contradiction

Prove that $\log_2 3$ is irrational.

Mathematical Induction

- A powerful proof technique in discrete (as opposed to continuous) math
- Systematic: provides a template for proving a wide range of properties

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Let P be a predicate on non-negative integers. If

- $P(0)$ is true, and
- $P(n)$ implies $P(n+1)$ for all nonnegative integers n, then
- conclude $P(m)$ is true for all nonnegative integers m

Induction (inference) rule: $$\frac{P(0), \forall n\ P(n) \rightarrow P(n+1)}{\forall m\ P(m)}$$

A Template for Induction Proofs

Never *omit any of these steps* in your proofs.

1. State that the proof uses induction.
   - For many proofs involving natural numbers, induction is on the number itself. But in other cases, it may be on another quantity, e.g., length of a string. In such cases, indicate the quantity on which induction is being carried out.
2. Define the induction hypothesis, namely, the predicate $P(n)$.
   - Often, P is the property you want to prove. But sometimes, you select a stronger property Q (i.e., Q implies P).
3. Establish the base case, i.e., show that $P(0)$ is true.
4. Establish the induction step, i.e., show that if $P(n)$ is true, then $P(n+1)$ holds too.
5. Invoke induction to conclude the proof.

Induction Proof Example: $\sum_{i=1}^{n} i$

1. **Proof:** is by induction on n.
2. *Induction Hypothesis:* Let $P(n) ::= \sum_{i=1}^{n} i = n(n+1)/2$.

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2. *Induction Hypothesis:* Let $P(n) ::= \sum_{i=1}^{n} i = n(n+1)/2$.
3. *Base Case:* For $n = 1$, $P(1)$ is $\sum_{i=1}^{1} i = 1 = 1(1+1)/2$. Thus, $P(1)$ holds.
4. *Induction Step:* Assume that $P(n)$ holds. Adding $n+1$ to both sides of $P(n)$, we get

$$
\begin{aligned}
\sum_{i=1}^{n+1} i &= \frac{n(n+1)}{2} + (n+1) \\
&= (n+1)\left(\frac{n}{2} + 1\right) && \text{(pulling out the common factor } n+1\text{)} \\
&= (n+1)\left(\frac{n+2}{2}\right) = \frac{(n+1)(n+2)}{2} && \text{(algebraic simplification)}
\end{aligned}
$$

5. Thus, we have established $P(n+1)$, thereby establishing $P(k)$ for all $k \geq 1$. ■

Induction Proof Example: $\forall n > 1 \; \sum_{i=1}^{n} 1/i^2 < 2 - 1/n$

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$$\sum_{i=1}^{n+1} 1/i^2 \quad < \quad 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$$

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$$\begin{aligned}\sum_{i=1}^{n+1} 1/i^2 &< 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \\ &= 2 - \frac{(n+1)^2 - n}{n(n+1)^2}\end{aligned}$$

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See https://faculty.math.illinois.edu/~hildebr/213/inductionsampler.pdf for more examples.

What is Wrong With This Proof?

Theorem

All horses have the same color

Base: Trivial, as there is a single horse.

Induction hypothesis: All sets of horses with n or fewer horses have the same color.

Induction Step: Consider a set of $h_1, h_2, \ldots, h_{n+1}$. By induction hypothesis:

$$\underbrace{h_1, h_2, \ldots, h_n}_{\text{same color}}, h_{n+1} \qquad\qquad h_1, \underbrace{h_2, \ldots, h_n, h_{n+1}}_{\text{same color}}$$

This obviously means that all $n+1$ horses have the same color!

Strong Induction

- *Key Point:* Makes the stronger assumption of $P(n), P(n-1), \ldots$
  - Contrast with simple induction, where we only assume $P(n)$.

(Strong) Induction Inference Rule

$$\frac{P(b),\ \forall n\ \left([\forall b \leq k \leq n\ \ P(k)] \rightarrow P(n+1)\right)}{\forall m \geq b\ P(m)}$$

- Secondary point: Base case can be for some small value b, not just zero
  - There can be more than one base case as well
- In some cases, noting the use of strong induction makes it easier to understand a proof
  - But the distinction is mostly insignificant

Proof by Strong Induction

The country Inductia, whose unit of currency is the Strong, has coins worth 3Sg (3 Strongs) and 5Sg. Although the Inductians have some trouble making small change like 4Sg or 7Sg, it turns out that they can collect coins to make change for any number that is at least 8 Strongs.

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3. *Base Cases:* For $n = 0$, $n = 1$ and $n = 2$, it is obvious that change can be made for 8Sg (3+5), 9Sg (3+3+3), and 10Sg (5+5). Thus $P(0)$, $P(1)$ and $P(2)$ hold.

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Since $P(n-2)$ holds, we know how to make change for $(n-2) + 8$ Sgs.

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If we add one 3Sg coin to this, we will have change for $(n+1) + 8$ Sgs.

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What does the proof say about making change when there is a severe shortage of 5Sg coins?

Proof by Strong Induction

Prime factorization theorem: Every integer $x > 1$ is a product of primes.

Stacking Game

- You begin with a stack of n chips, and end with n stacks of 1 chip each
- In each move of the game, you split one stack into two
  - If a stack of $a + b$ chips is spit into two stacks with a and b chips each, you get ab points.
- What strategy will maximize your winning?

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- In each move of the game, you split one stack into two
  - If a stack of $a + b$ chips is spit into two stacks with a and b chips each, you get ab points.
- What strategy will maximize your winning? Actually, the strategy does not matter!

Theorem

Every way of unstacking n blocks gives a score of $n(n-1)/2$ points.

Proof: is by induction on n.

(Strong) Induction Proof Example: Stacking Game

Induction Hypothesis: is that any way of unstacking k chips gives a score of $S(k) ::= k(k-1)/2$ points.

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Base Case: For $n = 1$, there are no moves left, so the score will be zero. This matches $S(1) = 1 \cdot (1-1)/2 = 0$, thus establishing the base.

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\begin{aligned}
&= r(n+1-r) + r(r-1)/2 + (n+1-r)(n-r)/2 \\
&= (nr + r - r^2) \; + \; (r^2 - r + n^2 - nr + n - r - nr + r^2)/2 && \text{(distributing multiplication)}
\end{aligned}
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Thus, no matter what the first move is (i.e., regardless of the value of r), we have shown that starting with $n+1$ chips, you end up with the score of $S(n+1)$, thus completing the inducation step. ∎

Proof by Induction

You are given a series of envelopes, respectively containing $1, 2, 4, 8, \ldots, 2^m$ dollars. Show that for any $0 \leq n < 2^{m+1}$ there is a selection of envelopes whose contents add up to exactly that number of dollars.

Fibonacci Numbers

- Arose out of population growth modeling work done by mathematician Fibonacci at the beginning of the 13th century.
- These numbers occur frequently in the natural world — number of petals in a flower, the number of growing tips in a tree, etc.

$$
\begin{aligned}
F(0) &= 0, \\
F(1) &= 1, \\
F(n) &= F(n-1) + F(n-2), \text{ for } n \geq 2.
\end{aligned}
$$

Induction Proof: Closed form solution for Fibonacci

Closed form Solution for $F(n)$

$$F(n) = \frac{p^n - q^n}{\sqrt{5}} \text{ where } p = \frac{1+\sqrt{5}}{2} \text{ and } q = \frac{1-\sqrt{5}}{2}$$

Induction Proof: Closed form solution for Fibonacci

Note that p and q are the roots of the quadratic equation $x^2 - x - 1 = 0$. So:

- $p^2 = p + 1$, and
- $q^2 = q + 1$

We will use these facts in our proof.

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$$F(n+1) \quad = \quad 1/\sqrt{5}((p^n - q^n) + (p^{n-1} - q^{n-1})) \qquad \text{(by induction hypothesis and then regrouping)}$$

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Induction Vs Proof-By-Contradiction