Counting (Textbook §14.1 to §14.9)

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Counting and Mappings

- If there a bijection $f: A \longrightarrow B$ then $|A| = |B|$
- More generally, if there is a mapping $g: A \longrightarrow B$

  with "$=1$ arrow **out**" and "$=k$ arrow **in**" properties

  then $|B| = |A|/k$ (Division Rule)

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  then $|B| = |A|/k$ (Division Rule)
- Additionally, we use what we already know about sizes of sets:
  - If $|A| = n$ then $|\mathbf{P}(A)| = 2^n$
  - If A and B are disjoint, $|A \cup B| = |A| + |B|$ (Sum Rule)
  - Size of $A \times B$ is $|A| \times |B|$ (Product Rule)

Sum Rule

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Sum Rule

If $P_1, P_2, \ldots, P_n$ are disjoint sets, then

$$|P_1 \cup P_2 \cup \cdots \cup P_n| = |P_1| + |P_2| + \cdots + |P_n|$$

- Since students don't take the two courses simultaneously, the two sets are disjoint, so we can apply the sum rule.
- The same rule cannot be applied to find the combined number of students across CSE 150 and AMS 210.
  - They may be taken simultaneously, so the sets overlap

Donut Selection I

- *World's Best Donut Shop* (WBDS) is so famous that everyone wants to buy a donut there.
- To maximize the number of customers serviced, the shop limits each customer to just one Donut.
- To ensure a unique customer experience, their Chef insists that every donut be distinct.
- If WBDS offers a choice of
  - 10 possible flavors
  - 8 possible fillings
  - 5 possible toppings
  - 8 possible sprinkles
- How many customers can WBDS serve in a day?

Product Rule

- The donut problem can be reduced to one of counting sequences
  - A donut is characterized by the sequence (F, I, T, S) representing the flavor, filling, topping and sprinkle choices

Product Rule

If $P_1, P_2, \ldots, P_n$ are finite sets, then

$$|P_1 \times P_2 \times \cdots \times P_n| = |P_1| \cdot |P_2| \cdot \cdots \cdot |P_n|$$

- Thus, the number of possible donuts = $|F| \cdot |I| \cdot |T| \cdot |S|$ = 10*8*5*8 = 3200

Key Assumption: The choices are independent of each other

Generalized Product Rule: Pieces on a Chessboard

- How many ways are there to arrange a pawn, a knight and a rook on a chessboard such that no two pieces occupy the same row or column?
  - Let us represent these positions as the sequence $(r_p, c_p, r_k, c_k, r_r, c_r)$.

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- The pawn can be in any of the 8 rows and columns, so r_p and c_p have 8 possible values.
- The knight can be in one of the remaining 7 rows/columns, so r_k and c_k have 7 possible values.
- The rook can be in one of the 6 rows and columns that are free after placing the pawn and the knight, so r_r and c_r have 6 possible values
- The total number of positions is hence $8 \cdot 8 \cdot 7 \cdot 7 \cdot 6 \cdot 6 = 112,896$

Permutations

- A permutation of a set S is a sequence that contains every element of S exactly once
- How many distinct permutations of S are there, if $|S| = n$?
  - The First element of the sequence can be any of n elements
  - Second element can be one of the remaining $n-1$ elements
  - Third element can be one of the remaining $n-2$ elements
  - Fourth element can be one of the remaining $n-3$ elements
  - Continuing on, we arrive at $n!$

Words of length r over an alphabet of size n

- If letters can be repeated:
  - Every letter can be one of n letters in the alphabet.
  - So, the total number of possibilities is n^r.
- If letters cannot be repeated:
  - Then we are asking for ${}^nP_r$, r-length permutations of n letters.
  - The first letter can be chosen in n ways
  - The second letter can be chosen from the remaining $n - 1$ letters
  - The k'th letter can be chosen from $n - k + 1$ letters,
    - i.e., after leaving out the letters already used in the preceding $k - 1$ letters
  - So, the number is:

$$n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

Award Distribution

- In how many ways can we distribute awards $A_1, \ldots, A_r$ to n persons?
  - We can represent the award as a sequence $(p_1, \ldots, p_r)$, where the p_i denotes the person winning the award A_i.
  - Thus, the number of possibilities is n^r
- But what if each person can win only one award?
  - For the ith award, the $i - 1$ persons that won awards 1 through $i - 1$ are *not* eligible
- Using this reasoning, we can calculate the number of possibilities as:

$$n \cdot (n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!} = {}^{n}P_r$$

Division Rule

- How many ways are there to arrange two identical rooks on a chessboard such that they occupy distinct rows and columns?
  - If the rooks are distinct, e.g., black and white, the answer is $8 \cdot 8 \cdot 7 \cdot 7 = 3136$
  - What if the rooks are of the same color?
    - Note that the position (r_1, c_1, r_2, c_2) is indistinguishable from (r_2, c_2, r_1, c_1) because the two rooks are identical

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- By applying division rule, we arrive at the correct number for identical rooks: $3136/2 = 1568$

Knights around a Circular Table

- How many distinct ways can n knights be seated around a circular table?
  - Two seatings are considered the same if the two knights sitting next to each knight remains the same in both seatings.
- Let there be n knights and m seats around a table. A seating can be captured by listing the seats in order $(k_1, k_2, \ldots, k_m)$, which translates to $n!/(n-m)!$ seatings.
- But many sequences represent the "same" seating arrangement. The first row lists the participants in clockwise order, starting from one of the m seats. The corresponding anti-clockwise order listing is shown in the second row.

$$\begin{array}{ccccc} (k_1, k_2, \ldots, k_m) & (k_2, \ldots, k_m, k_1) & (k_3, \ldots, k_m, k_1, k_2) & \cdots & (k_m, k_1, \ldots, k_{m-1}) \\ (k_1, k_m, k_{m-1} \ldots, k_2) & (k_2, k_1, k_m, \ldots k_3) & (k_3, k_2, \ldots k_4) & \cdots & (k_m, k_{m-1}, k_{m-2}, \ldots, k_1) \end{array}$$

- Using division rule, we obtain the number of distinct seatings as $\frac{n!}{(n-m)!} \cdot \frac{1}{2m} = \frac{n!}{2(n-m)!m}$

Combinations aka Counting Subsets

- How many distinct 5-card poker hands can be dealt from a 52-card deck?
- How many ways can I select 3 toppings for my pizza from the 10 available toppings?

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Subset Rule

A set of size n has $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ distinct subsets of size m

- Start with a m-length permutations of the form $(i_1, i_2, \ldots, i_m)$
  - By the permutation rule, there are $\frac{n!}{(n-m)!}$ such sequences
- Apply division rule to collapse sequences corresponding to the same set:
  - There are $m!$ permutations of an m-element set
- Thus, there are $\frac{n!}{(n-m)!m!} = \binom{n}{m}$ distinct subsets (aka combinations)

Donut Selection II

- Despite its unparalleled reputation, WBDS management is unhappy with the sales volume. They propose selling one box of dozen donuts per customer.
- Chef's agrees to sell the same donut to multiple customers, as long as no two customers get the exact some dozen.
- How many donuts can they sell in a day?

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- Chef's agrees to sell the same donut to multiple customers, as long as no two customers get the exact some dozen.
- How many donuts can they sell in a day?
  - Problem: How many distinct subsets of 12 can be drawn from 3200?

$$\binom{3200}{12} \approx 2.36 \text{ Decillion!!! (i.e., } 2.36 \times 10^{33}\text{)}$$

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    - So, WBDS can sell up to $12 \cdot \binom{3200}{12} \approx 28$ decillion Donuts!
- What is the minimum donuts per box to be able to sell to everyone in the world?
  - $\binom{3200}{3} \approx 5.5$B will almost do!

Sequences of Subsets

- In how many ways can we split a set of size n into subsets of size k_1, k_2, ..., k_m?
- Each split is a collection of sets $\{A_1, A_2, \ldots, A_m\}$ such that $|A_i| = k_i$ and $\sum_{i=1}^{m} k_i = n$.
- Given a permutation of n elements, we can treat the first k_1 elements in this sequence as A_1, the next k_2 elements as A_2 and so on.
- However, the same elements of A_i are represented using $k_i!$ distinct permutations.
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  - So we apply division rule for A_1, ..., A_m:

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}$$

$\binom{n}{m}$ is called a *binomial coefficient*; $\binom{n}{k_1, k_2, \ldots, k_m}$ is called a *multinomial coefficient*

Defective Dollar Bills

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- Here, the complement set is one that contains *no* repeated digit.
  - If no digit is repeated, the number of possible dollar bills is $10!/(10-8)! = 10!/2$
  - If repetitions are permitted, number of possible bills is 10^8
- So, number of defective bills is $10^8 - 10!/2 = 98185600$.

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- So, number of defective bills is $10^8 - 10!/2 = 98185600$.
- Fraction of non-defective bills is $(10 \cdot 9 \cdot 8 \cdots 3)/10^8 = 1.81\%$
  - Over 49 of 50 possible dollar bills are defective!

Complement Rule: Additional Examples

- If you toss a fair coin n times, what is the likelihood there will be one or more heads?
  - Very easy to count sequences that contain no heads: there is exactly one!
  - It is also easy to total number of possible sequences: 2^n
  - With a fair coin, all sequences are equally likely, so the probability is $(2^n - 1)/2^n$

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  - With a fair coin, all sequences are equally likely, so the probability is $(2^n - 1)/2^n$
- In a group of five students, what is the likelihood of finding two students that were born on the same day of the week?
  - Easier to count instances where people were born on distinct days
    - $\binom{7}{5}$ ways for five students to be born on 5 distinct days.
  - 7^5 ways if there are no constraints.
  - So, the desired probability, assuming that students are equally likely to be born any day of the week, is $(7^5 - \binom{7}{5})/7^5 = 0.99875$

Summary of Basic Counting Rules

Product rule: $|P_1 \times P_2 \times \cdots \times P_n| = |P_1| \cdot |P_2| \cdot \cdots \cdot |P_n|$ *If choices are independent*

Sum rule: $|P_1 \cup P_2 \cup \cdots \cup P_n| = |P_1| + |P_2| + \cdots + |P_n|$ *If sets are disjoint*

Complement rule: $|A| = |U| - |\overline{A}|$

- Alternative form: If $A \subseteq S$ then $|A| = |S| - |S - A|$

Division rule: If $f : A \longrightarrow B$ is a k-to-1 onto function then $|A| = k \cdot |B|$.

Permutation rule: The number of r-length permutation of n elements is $\frac{n!}{(n-r)!}$.

Combinations/subset rule: A set of size n has $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ distinct subsets of size m.

Words from Repeated Letters: Bookkeeper

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- Consider all permutations, then use division rule to eliminate duplicates
- Total permutations = 10!
- Repetitions: 3 E's, 2 O'S, 2 K's

$$\frac{10!}{3!2!2!} = 151200$$

Poker Hands: Four of a Kind

- How many 5-card hands have all four suits of some rank?
  - $8\spadesuit 8\diamondsuit 8\heartsuit 8\clubsuit 3\heartsuit$
  - $2\spadesuit Q\spadesuit 2\diamondsuit 2\heartsuit 2\clubsuit$

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- First pick the repeating rank: 1 of 13 ways
- Next pick the fifth card: 1 of 48 ways
- Total: $13 \cdot 48 = 624$

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- Total: $13 \cdot 48 = 624$
- What is the probability of being dealt a four-of-a-kind hand?
  - Divide by number of possible 5-card hands

$$\frac{624}{\binom{52}{5}} = \frac{624}{2,598,960} = 0.024\%$$

Poker Hands: Full House

- A *Full House* is a hand with three cards of one rank and two of another rank.

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- A *Full House* is a hand with three cards of one rank and two of another rank.
- Choose the rank of three cards: 13 ways
- Choose the suits of three cards: $\binom{4}{3} = \binom{4}{1} = 4$
- Choose the rank of two cards: 12 ways
- Choose the suits of two cards: $\binom{4}{2} = 6$
- Apply the product rule to get $13 \cdot 4 \cdot 12 \cdot 6 = 3744$

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- Choose the rank of two cards: 12 ways
- Choose the suits of two cards: $\binom{4}{2} = 6$
- Apply the product rule to get $13 \cdot 4 \cdot 12 \cdot 6 = 3744$
- The probability is significantly higher than 4-of-a-kind, but still very low: 0.14%

Poker Hands: Non-repeating Ranks

- Approach 1

Poker Hands: Non-repeating Ranks

- Approach 1
  - Choose the 5 ranks in $\binom{13}{5}$ ways
  - Choose the suit of each card in 4 ways
  - Total: $\binom{13}{5} \cdot 4^5 = 1,317,888$ ways
- Approach 2

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- Approach 2
  - Count 5-card sequences with distinct ranks, then use division rule
    - Choose the first card in 52 ways, second card in 48 ways, and so on
    - Finally divide by 5! to get: $\frac{52 \cdot 48 \cdot 44 \cdot 40 \cdot 36}{5!} = 1,317,888$ ways
- Dividing by the total number of hands, we get

$$\frac{1,317,888}{\binom{52}{5}} = \frac{1,317,888}{2,598,960} = 0.5071$$

Poker Hands: Two Pairs

- Two cards of one rank, two cards of another rank, a fifth card of different rank

Poker Hands: Two Pairs

- Two cards of one rank, two cards of another rank, a fifth card of different rank
- First pair can be chosen in
$$\binom{13}{1}\binom{4}{2} = 78 \text{ ways}$$
- Second pair can be chosen in
$$\binom{12}{1}\binom{4}{2} = 72 \text{ ways}$$
- Fifth card can be chosen in 44 ways (leave out all suits of the two chosen ranks)
- Since we can't distinguish between the first and second pairs, divide by 2!
$$\frac{78 \cdot 72 \cdot 44}{2} = 123,552 \text{ ways}$$

Poker Hands: Two Pairs — Alternate Method

- Pick the rank of the two pairs: $\binom{13}{2}$ ways
- Pick the suit of the lower-ranked pair: $\binom{4}{2}$ ways
- Pick the suit of the higher-ranked pair: $\binom{4}{2}$ ways
- Pick the rank of extra card: 11 ways
- Pick the suit of the extra card: 4 ways
- Total:

$$\binom{13}{2} \cdot \binom{4}{2}^2 \cdot 11 \cdot 4 = 123,552 \text{ ways}$$

Poker Hands: Every Suit

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- Key point: the choice of the fifth card is not entirely independent of the first four.
- For the first four cards, the suites are fixed, so there is a choice only for the ranks:

$$13 \cdot 13 \cdot 13 \cdot 13 = 13^4$$

- The last card can be any of the remaining 48
  - But if we switch the last card with the other card that has the same suit, we would have counted that as a separate hand.
  - So we need to divide in the end by 2

$$\frac{13^4 \cdot 48}{2} = 685,464$$

Poker Hands: At least 4 Ranks

- Break into two cases:
  - Exactly 5 ranks (already done): $1,317,888$ ways
  - Exactly 4 ranks

Poker Hands: At least 4 Ranks

- Break into two cases:
  - Exactly 5 ranks (already done): $1,317,888$ ways
  - Exactly 4 ranks
    - Choose the 4 ranks in $\binom{13}{4}$ ways
    - Choose which rank to repeat: 4 ways
    - Choose the suit of 3 non-repeating ranks: 4^3 ways
    - Choose the suits of repeating rank in $\binom{4}{2}$ ways
    - Total: $\binom{13}{4} \cdot 4 \cdot 4^3 \cdot \binom{4}{2} = 1,098,240$ ways
- Total across two cases: $1,317,888 + 1,098,240 = 2416128$ ways

Number of Relations

How many binary relations from X to Y are there? State your answer in terms of the cardinalities of X and Y.

Number of Bijections

Let $|X| = n$. How many bijections are there from X to X?

Binary Strings with Exactly k Ones

- How many n-bit sequences have exactly k ones?

Binary Strings with Exactly k Ones

- How many n-bit sequences have exactly k ones?
- You need to select a subset of k positions that will be occupied by 1's, while all other positions will be zeroes.
- So, the number is

$$\binom{n}{k}$$

Donut Selection III

- To further increase profits, WBDS management wants to streamline production by cutting down the number of distinct donut types to 10.
- The Chef continues to insist on a unique box of dozen for each customer.
- How many customers can WBDS serve per day with these new rules?

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- The Chef continues to insist on a unique box of dozen for each customer.
- How many customers can WBDS serve per day with these new rules?
- You can map it to a bit strings problem!
  - The 0's represent the donuts, 1's represent the boundary between donut types.
  - We need 9 boundaries for 10 donut types, and 12 zeroes for 12 donuts.
- Thus, the total number of boxes is

$$\binom{21}{9} \approx 300K$$

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- How about the general case of a box of n Donuts drawn from m types?
  - We need $m-1$ ones and n zeroes, so the number is $\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$

Donut Selection: Variants

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- Approach 1: Map to binary strings with 2 zeroes and 9 ones
  - $\binom{11}{9} = \binom{11}{2} = \frac{11 \cdot 10}{2} = 55$

Donut Selection: Variants

- What if you add a requirement that there be at least one donut of each type?
  - Put one donut of each type into the box, then count the ways to choose the other two
- Approach 1: Map to binary strings with 2 zeroes and 9 ones
  - $\binom{11}{9} = \binom{11}{2} = \frac{11 \cdot 10}{2} = 55$
- Approach 2: Decompose into union of disjoint sets
  - The last two donuts can have (i) the same type, or (ii) distinct types
    - There are 10 choices for (i), and
    - $\binom{10}{2} = \frac{10 \cdot 9}{2} = 45$ choices for (ii)
  - The total is again 55.

Donut Selection: Variants

- What if you add a requirement that there be at least one donut of each type?
  - Put one donut of each type into the box, then count the ways to choose the other two
- Approach 1: Map to binary strings with 2 zeroes and 9 ones
  - $\binom{11}{9} = \binom{11}{2} = \frac{11 \cdot 10}{2} = 55$
- Approach 2: Decompose into union of disjoint sets
  - The last two donuts can have (i) the same type, or (ii) distinct types
    - There are 10 choices for (i), and
    - $\binom{10}{2} = \frac{10 \cdot 9}{2} = 45$ choices for (ii)
  - The total is again 55.

> Often, there are multiple ways of counting
> - but all should yield the same answer: use this to check your approach.

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- There are 6 books on a shelf. You want to choose 3 books such that no consecutive books are chosen.

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- For the bit string, this constraint becomes:
  - "Ten 1-bits and fifteen zero bits with all but the last 1 to be followed by a 0"

Choosing Books (Continued)

- Consider "Ten 1-bits and fifteen zero bits with all but the last 1 to be followed by a 0"
- The constraint about 0-bit following a 1-bit is precisely captured by "taping" a zero to all but the last 1.
  - Since there are nine such 1's, this effectively reduces the number of positions by 9

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- Thus, we need to count bit strings with $25 - 9 = 16$ positions with ten 1's.
  - So, the number of possible selections is $\binom{16}{10} = 8008$
- Can we generalize to m books out of n?
  - The "taping" step above reduces the number of positions from n to $n - m + 1$
  - So, the number is $\binom{n-m+1}{m}$

Partitioning an Integer

Let $S_{n,k}$ be the possible non-negative integer solutions to the inequality

$$x_1 + x_2 + \cdots + x_k = n$$

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This is just the Donut problem!

- Make a box of n Donuts, drawing from k different types.
- So, the solution is
$$S_{n,k} = \binom{n+k-1}{n}$$

Partitioning an Integer

Let $I_{n,k}$ be the possible non-negative integer solutions to the inequality

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Can be reduced the equality case from previous slide:

$$x_1 + x_2 + \cdots + x_k + x_{k+1} = n$$

For any value $n' \leq n$ such that $x_1 + x_2 + \cdots + x_k = n'$, $x_{k+1} = n - n'$

- Thus there is a bijection between the equality and inequality formulations.
- So, the solution is

$$I_{n,k} = \binom{n+k}{n}$$

Increasing Sequences

Let $L_{n,k}$ be the length of k weakly increasing sequences of non-negative integers, i.e.,

$$y_1 \le y_2 \le \cdots \le y_k \le n$$

How many such sequences are there?

This is the same problem from the last slide!

- Let y_i denote the sum of x_1 through x_i

$$y_i = \sum_{j=1}^{i} x_j$$

Pigeonhole Principle

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Formal Version

If $|A| > |B|$ then every total function $f\colon A \longrightarrow B$ maps at least two different elements of A to an element of B

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Generalized Pigeonhole Principle

If $|A| > k \cdot |B|$ then every total function $f\colon A \longrightarrow B$ maps at least $k+1$ different elements of A to an element of B

Subset With Same Sum

- Suppose that we generate 100 random numbers with 25 digits:

  0020480135385502964448038

  5763257331083479647409398

  0489445991866915676240992

  ⋮

- Will there be two subsets of these 25-digit numbers that add up to the same value?

Union of Overlapping Sets: Inclusion-Exclusion Principle

- So far, we studied union of disjoint sets, where:

$$|S_1 \cup S_2| = |S_1| + |S_2|$$

- What happens when the sets overlap?

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Union of Two Sets

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$$

Union of Three Sets

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$$|S_1 \cup S_2 \cup S_3| = (|S_1| + |S_2| + |S_3|) \ - \ (|S_1 \cap S_2| + |S_2 \cap S_3| + |S_3 \cap S_1|) \ + \ |S_1 \cap S_2 \cap S_3|$$

Union of n sets

Inclusion-Exclusion for n Sets

$$\left|\bigcup_{i=1}^{n} S_i\right| = \sum_{i=1}^{n} |S_i| - \sum_{1\le i<j\le n} |S_i \cap S_j| + \sum_{1\le i<j<k\le n} |S_i \cap S_j \cap S_k| \cdots (-1)^{n-1} \left|\bigcap_{i=1}^{n} S_i\right|$$

Or, more compactly:

$$\left|\bigcup_{i=1}^{n} S_i\right| = \sum_{I\in\wp(\{1,2,\ldots,n\})} (-1)^{|I|+1} \left|\bigcap_{i\in I} S_i\right|$$

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- Permutations containing 42: "Fuse" 4 and 2 together, treat as if they are a single symbol. Now we have 9 symbols, for $9!$ permutations
- Permutations containing 04 and 60 are also $9!$ each
- Permutations containing two of these pairs would be $8!$
- Permutations containing all three would be $7!$
- Using inclusion-exclusion principle, we arrive at

$$3 \cdot 9! - 3 \cdot 8! + 7!$$

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- How about sequences with 42, 62, or 60?

Numbers Containing Certain Digits

- How many numbers between 1 and 100 contain a 5 in them?

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Numbers Containing Certain Digits

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- How many numbers between 1 and 1000 contain a 5 in them?
- How many numbers between 1 and 1B contain a 5 in them?
- *Inclusion-exclusion can be cumbersome to use.* You should first check if other rules (e.g., complement) are applicable.

Summary of Counting Rules

Product rule: $|P_1 \times P_2 \times \cdots \times P_n| = |P_1| \cdot |P_2| \cdot \cdots \cdot |P_n|$ *If choices are independent*

Sum rule: $|P_1 \cup P_2 \cup \cdots \cup P_n| = |P_1| + |P_2| + \cdots + |P_n|$ *If sets are disjoint*

Complement rule: $|A| = |U| - |\overline{A}|$ (Alternative form: If $A \subseteq S$ then $|A| = |S| - |S - A|$)

Division rule: If $f : A \longrightarrow B$ is a k-to-1 onto function then $|A| = k \cdot |B|$.

Permutation rule: The number of r-length permutation of n elements is $\frac{n!}{(n-r)!}$.

Combinations/subset rule: A set of size n has $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ distinct subsets of size m.

Bijection with bit strings.

Pigeon Hole Principle.

Inclusion-Exclusion Principle.