

Counting (Textbook §14.1 to §14.9)

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Donut Selection I

- *World's Best Donut Shop* (WBDS) is so famous that everyone wants to buy a donut there.
- To maximize the number of customers serviced, the shop limits each customer to just one Donut.
- To ensure a unique customer experience, their Chef insists that every donut be distinct.
- If WBDS offers a choice of
 - 10 possible flavors
 - 8 possible fillings
 - 5 possible toppings
 - 8 possible sprinkles
- How many customers can WBDS serve in a day?

Product Rule

- The donut problem can be reduced to one of counting sequences
 - A donut is characterized by the sequence (F, I, T, S) representing the flavor, filling, topping and sprinkle choices

Product Rule

If P_1, P_2, \dots, P_n are finite sets, then

$$|P_1 \times P_2 \times \dots \times P_n| = |P_1| \cdot |P_2| \cdot \dots \cdot |P_n|$$

- Thus, the number of possible donuts = $|F| \cdot |I| \cdot |T| \cdot |S| = 10 \cdot 8 \cdot 5 \cdot 8 = 3200$

Key Assumption: The choices are independent of each other

Permutations

- A permutation of a set S is a sequence that contains every element of S exactly once
- How many distinct permutations of S are there, if $|S| = n$?
 - The First element of the sequence can be any of n elements
 - Second element can be one of the remaining $n-1$ elements
 - Third element can be one of the remaining $n-2$ elements
 - Fourth element can be one of the remaining $n-3$ elements
 - Continuing on, we arrive at $n!$

Words of length r over an alphabet of size n

- If letters can be repeated:
 - Every letter can be one of n letters in the alphabet.
 - So, the total number of possibilities is n^r .
- If letters cannot be repeated:
 - Then we are asking for ${}^n P_r$, r -length permutations of n letters.
 - The first letter can be chosen in n ways
 - The second letter can be chosen from the remaining $n - 1$ letters
 - The k 'th letter can be chosen from $n - k + 1$ letters,
 - i.e., after leaving out the letters already used in the preceding $k - 1$ letters
 - So, the number is:

$$n \cdot (n - 1) \cdots (n - r + 1) = \frac{n!}{(n - r)!}$$

Defective Dollar Bills

- Let us call a dollar bill *defective* if any digit in its 8-digit serial number is repeated.
 - How many defective dollar bills are possible?
- Often, it is easier to count the *complement* of a set:

Complement Rule

$$|A| = |U| - |\bar{A}|$$

- Here, the complement set is one that contains *no* repeated digit.
 - If no digit is repeated, the number of possible dollar bills is $10!/(10 - 8)! = 10!/2$
 - If repetitions are permitted, number of possible bills is 10^8
- So, number of defective bills is $10^8 - 10!/2 = 98185600$.
- Fraction of non-defective bills is $(10 \cdot 9 \cdot 8 \cdots 3)/10^8 = 1.81\%$
 - Over 49 of 50 possible dollar bills are defective!

A Coin Toss Problem

- If you toss a fair coin n times, what is the likelihood there will be one or more heads?
 - Very easy to count sequences that contain no heads: there is exactly one!
 - It is also easy to total number of possible sequences: 2^n
 - With a fair coin, all sequences are equally likely, so the probability is $(2^n - 1)/2^n$
- In a group of five students, what is the likelihood of finding two students that were born on the same day of the week?
 - Easier to count instances where people were born on distinct days
 - $\binom{7}{5}$ ways for five students to be born on 5 distinct days.
 - 7^5 ways if there are no constraints.
 - So, the desired probability, assuming that students are equally likely to be born any day of the week, is $(7^5 - \binom{7}{5})/7^5 = 0.99875$

Disjoint Sum Rule

- If there are 60 students enrolled in CSE 150 and 50 students enrolled in CSE 350, how many students are enrolled across the two courses?

Sum Rule

If P_1, P_2, \dots, P_n are disjoint sets, then

$$|P_1 \cup P_2 \cup \dots \cup P_n| = |P_1| + |P_2| + \dots + |P_n|$$

- Since students don't take the two courses simultaneously, the two sets are disjoint, so we can apply the sum rule.
- The same rule cannot be applied to find the combined number of students across CSE 150 and AMS 210.
 - They may be taken simultaneously, so the sets overlap

Numbers Containing Certain Digits

- How many numbers between 1 and 100 contain a 5 in them?

Union of Overlapping Sets: Inclusion-Exclusion Principle

Union of Two Sets

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$$

Union of Three Sets

Union of Three Sets

$$|S_1 \cup S_2 \cup S_3| = (|S_1| + |S_2| + |S_3|) - (|S_1 \cap S_2| + |S_2 \cap S_3| + |S_3 \cap S_1|) + |S_1 \cap S_2 \cap S_3|$$

Numbers Containing Certain Digits

- How many numbers between 1 and 1000 contain a 5 in them?

Sequences with 42, 04 or 60

- How many permutations of $\{0, 1, 2, \dots, 9\}$ contain a 42, 04 or 60?
- Permutations containing 42: “Fuse” 4 and 2 together, treat as if they are a single symbol. Now we have 9 symbols, for $9!$ permutations
- Permutations containing 04 and 60 are also $9!$ each
- Permutations containing two of these pairs would be $8!$
- Permutations containing all three would be $7!$
- Using inclusion-exclusion principle, we arrive at

$$3 \cdot 9! - 3 \cdot 8! + 7!$$

Union of n sets

Inclusion-Exclusion for n Sets

$$\left| \bigcup_{i=1}^n S_i \right| = \sum_{i=1}^n |S_i| - \sum_{1 \leq i < j \leq n} |S_i \cap S_j| + \sum_{1 \leq i < j < k \leq n} |S_i \cap S_j \cap S_k| \cdots (-1)^{n-1} \left| \bigcap_{i=1}^n S_i \right|$$

Or, more compactly:

$$\left| \bigcup_{i=1}^n S_i \right| = \sum_{I \in \wp(\{1,2,\dots,n\})} (-1)^{|I|+1} \left| \bigcap_{i \in I} S_i \right|$$

Numbers Containing Certain Digits

- How many numbers between 1 and 10^8 contain a 5 in them?
- *Inclusion-exclusion can be cumbersome to use.* You should first check if other rules (e.g., complement) are applicable.

Division Rule

- How many ways are there to arrange two identical rooks on a chessboard such that they occupy distinct rows and columns?
- If the rooks are distinct, e.g., black and white, the answer is $8 \cdot 8 \cdot 7 \cdot 7 = 3136$
- What if the rooks are of the same color?
 - Note that the position (r_1, c_1, r_2, c_2) is indistinguishable from (r_2, c_2, r_1, c_1) because the two rooks are identical

Division Rule

If $f: A \longrightarrow B$ is a k -to-1 onto function then $|A| = k \cdot |B|$
(Such a function has $=1$ arrow **out** and $=k$ arrow **in** properties.)

- By applying division rule, we arrive at the correct number for identical rooks:
 $3136/2 = 1568$

Combinations aka Counting Subsets

- How many distinct 5-card poker hands can be dealt from a 52-card deck?
- How many ways can I select 3 toppings for my pizza from the 10 available toppings?

Subset Rule

A set of size n has $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ distinct subsets of size m

- Start with a m -length permutations of the form (i_1, i_2, \dots, i_m)
 - By the permutation rule, there are $\frac{n!}{(n-m)!}$ such sequences
- Apply division rule to collapse sequences corresponding to the same set:
 - There are $m!$ permutations of an m -element set
- Thus, there are $\frac{n!}{(n-m)!m!} = \binom{n}{m}$ distinct subsets (aka combinations)

Sequences of Subsets

- In how many ways can we split a set of size n into subsets of size k_1, k_2, \dots, k_m ?
- Each split is a collection of sets $\{A_1, A_2, \dots, A_m\}$ such that $|A_i| = k_i$ and $\sum_{i=1}^m k_i = n$.
- Given a permutation of n elements, we can treat the first k_1 elements in this sequence as A_1 , the next k_2 elements as A_2 and so on.
- However, the same elements of A_i are represented using $k_i!$ distinct permutations.
 - So we apply division rule for A_1, \dots, A_m :

$$\binom{n}{k_1, k_2, \dots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}$$

$\binom{n}{m}$ is called a *binomial coefficient*; $\binom{n}{k_1, k_2, \dots, k_m}$ is called a *multinomial coefficient*

Summary of Basic Counting Rules

Sum rule:

- *Simple sum for disjoint sets:* $|P_1 \cup P_2 \cup \dots \cup P_n| = |P_1| + |P_2| + \dots + |P_n|$.
- *Inclusion-Exclusion principle* to be applied for overlapping sets.

Difference (Complement) rule: $|A| = |U| - |\overline{A}|$

- Alternative form: If $A \subseteq S$ then $|A| = |S| - |S - A|$

Product rule: $|P_1 \times P_2 \times \dots \times P_n| = |P_1| \cdot |P_2| \cdot \dots \cdot |P_n|$, if choices are independent

- *Permutation rule:* The number of r -length permutation of n elements is $\frac{n!}{(n-r)!}$.

Division rule: If $f: A \longrightarrow B$ is a k -to-1 onto function then $|A| = k \cdot |B|$.

- *Combinations rule:* A set of size n has $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ distinct subsets of size m .
- *Sequences of subsets rule:* $\binom{n}{k_1, k_2, \dots, k_m} = \frac{n!}{k_1! k_2! \dots k_m!}$

Words from Repeated Letters: Bookkeeper

- How many distinct words can be formed by permuting letters in the word Bookkeeper?
- Consider all permutations, then use division rule to eliminate duplicates
- Total permutations = $10!$
- Repetitions: 3 E's, 2 O's, 2 K's

$$\frac{10!}{3!2!2!} = 151200$$

Donut Selection II

- Despite its unparalleled reputation, WBDS management is unhappy with the sales volume. They propose selling one box of dozen donuts per customer.
- Chef's agrees to sell the same donut to multiple customers, as long as (a) no donut is repeated in a box, and (b) no two customers get the exact same dozen.
- How many donuts can they sell in a day?

- Problem: How many distinct subsets of 12 can be drawn from 3200?

$$\binom{3200}{12} \approx 2.36 \text{ Decillion!!! (i.e., } 2.36 \times 10^{33}\text{)}$$

- So, WBDS can sell up to $12 \cdot \binom{3200}{12} \approx 28 \text{ decillion Donuts!}$
 - What is the minimum donuts per box to be able to sell to everyone in the world?
 - $\binom{3200}{3} \approx 5.5\text{B}$ will almost do!

Poker Hands: Four of a Kind

- How many 5-card hands have all four suits of same rank?

- $8\spadesuit 8\diamondsuit 8\heartsuit 8\clubsuit 3\heartsuit$

- $2\spadesuit Q\spadesuit 2\diamondsuit 2\heartsuit 2\clubsuit$

- First pick the repeating rank: 1 of 13 ways
- Next pick the fifth card: 1 of 48 ways
- Total: $13 \cdot 48 = 624$
- What is the probability of being dealt a four-of-a-kind hand?
 - Divide by number of possible 5-card hands

$$\frac{624}{\binom{52}{5}} = \frac{624}{2,598,960} = 0.024\%$$

Poker Hands: Non-repeating Ranks

- Approach 1
 - Choose the 5 ranks in $\binom{13}{5}$ ways
 - Choose the suit of each card in 4 ways
 - Total: $\binom{13}{5} \cdot 4^5 = 1,317,888$ ways
- Approach 2
 - Count 5-card sequences with distinct ranks, then use division rule
 - Choose the first card in 52 ways, second card in 48 ways, and so on
 - Finally divide by $5!$ to get: $\frac{52 \cdot 48 \cdot 44 \cdot 40 \cdot 36}{5!} = 1,317,888$ ways

Poker Hands: Two Pairs

- Two cards of one rank, two cards of another rank, a fifth card of different rank
- First pair can be chosen in

$$\binom{13}{1} \binom{4}{2} = 78 \text{ ways}$$

- Second pair can be chosen in

$$\binom{12}{1} \binom{4}{2} = 72 \text{ ways}$$

- Fifth card can be chosen in 44 ways (leave out all suits of the two chosen ranks)
- Since we can't distinguish between the first and second pairs, divide by 2!

$$\frac{78 \cdot 72 \cdot 44}{2} = 123,552 \text{ ways}$$

Poker Hands: Two Pairs — Alternate Method

- Pick the rank of the two pairs: $\binom{13}{2}$ ways
- Pick the suit of the lower-ranked pair: $\binom{4}{2}$ ways
- Pick the suit of the higher-ranked pair: $\binom{4}{2}$ ways
- Pick the rank of extra card: 11 ways
- Pick the suit of the extra card: 4 ways

- Total:

$$\binom{13}{2} \cdot \binom{4}{2}^2 \cdot 11 \cdot 4 = 123,552 \text{ ways}$$

Number of Relations

How many binary relations from X to Y are there? State your answer in terms of the cardinalities of X and Y .

Number of Functions

How many binary functions from X to Y are there? State your answer in terms of the cardinalities of X and Y .

Number of Bijections

Let $|X| = |Y| = n$. How many bijections are there from X to Y ?

Binary Strings with Exactly k Ones

- How many n -bit sequences have exactly k ones?
- You need to select a subset of k positions that will be occupied by 1's, while all other positions will be zeroes.
- So, the number is

$$\binom{n}{k}$$

Donut Selection III

- To further increase profits, WBDS management wants to streamline production by cutting down the number of distinct donut types to 10.
- The Chef continues to insist on a unique box of dozen for each customer.
- How many customers can WBDS serve per day with these new rules?
- You can map it to a bit strings problem!
 - The 0's represent the donuts, 1's represent the boundary between donut types.
 - We need 9 boundaries for 10 donut types, and 12 zeroes for 12 donuts.
- Thus, the total number of boxes is

$$\binom{21}{9} \approx 300K$$

Donut Selection: Variants

- What if you add a requirement that there be at least one donut of each type?
 - Put one donut of each type into the box, then count the ways to choose the other two
- Approach 1: Map to binary strings with 2 zeroes and 9 ones
 - $\binom{11}{9} = \binom{11}{2} = \frac{11 \cdot 10}{2} = 55$
- Approach 2: Decompose into union of disjoint sets
 - The last two donuts can have (i) the same type, or (ii) distinct types
 - There are 10 choices for (i), and
 - $\binom{10}{2} = \frac{10 \cdot 9}{2} = 45$ choices for (ii)
 - The total is again 55.

Often, there are multiple ways of counting

- but all should yield the same answer: use this to check your approach.

Partitioning an Integer

Let $S_{n,k}$ be the possible non-negative integer solutions to the equation

$$x_1 + x_2 + \cdots + x_k = n$$

How many solutions are there?

This is just the Donut problem!

- Make a box of n Donuts, drawing from k different types.
- So, the solution is

$$S_{n,k} = \binom{n+k-1}{n}$$

Partitioning an Integer

Let $I_{n,k}$ be the possible non-negative integer solutions to the inequality

$$x_1 + x_2 + \cdots + x_k \leq n$$

How many solutions are there?

Can be reduced the equality case from previous slide:

$$x_1 + x_2 + \cdots + x_k + x_{k+1} = n$$

- So, the solution is

$$I_{n,k} = \binom{n+k}{n}$$

Increasing Sequences

Let $L_{n,k}$ be the length of k weakly increasing sequences of non-negative integers, i.e.,

$$y_1 \leq y_2 \leq \cdots \leq y_k \leq n$$

How many such sequences are there?

This is the same problem from the last slide!

- Let y_i denote the sum of x_1 through x_i

$$y_i = \sum_{j=1}^i x_j$$

Pigeonhole Principle

- A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?

Simple Version

If there are more pigeons than the holes they occupy, then there must be at least two pigeons in some hole

Formal Version

If $|A| > |B|$ then every total function $f: A \longrightarrow B$ maps at least two different elements of A to an element of B

Hairs on Heads

- How many people riding the NYC subway in a day have the same number of hairs on their heads?
- Let us say that each head has a maximum of 100K hairs
- NYC daily subway ridership is about 3.6M

Generalized Pigeonhole Principle

If $|A| > k \cdot |B|$ then every total function $f: A \rightarrow B$ maps at least $k + 1$ different elements of A to an element of B

Subset With Same Sum

- Suppose that we generate 100 random numbers with 25 digits:

0020480135385502964448038

5763257331083479647409398

0489445991866915676240992

⋮

- Will there be two subsets of these 25-digit numbers that add up to the same value?

Choosing Books

- There are 6 books on a shelf. You want to choose 3 books such that no consecutive books are chosen.
- What if there are 25 books on the shelf, and you want to choose 10?
- Selection problems like this can often be mapped to bit strings:
 - Use 25 bits to represent 25 books, with 1's for selected books and 0's for unselected ones
- How to incorporate the constraints?
 - Select 10 books: There should be exactly ten 1-bits
 - No two consecutive: Can be simplified to “book following a selected book should be skipped.”
 - For the last selected book, “skip the following book” is redundant
 - We are done with all selections, so we are not going to select any more books
- For the bit string, this constraint becomes:
 - “Ten 1-bits and fifteen zero bits with all but the last 1 to be followed by a 0”

Choosing Books (Continued)

- Consider “Ten 1-bits and fifteen zero bits with all but the last 1 to be followed by a 0”
- The constraint about 0-bit following a 1-bit is precisely captured by “taping” a zero to all but the last 1.
 - Since there are nine such 1’s, this effectively reduces the number of positions by 9
- Thus, we need to count bit strings with $25 - 9 = 16$ positions with ten 1’s.
 - So, the number of possible selections is $\binom{16}{10} = 8008$
- Can we generalize to m books out of n ?
 - The “taping” step above reduces the number of positions from n to $n - m + 1$
 - So, the number is $\binom{n-m+1}{m}$

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Bijection with bit strings.

Pigeon Hole Principle.