Counting (Textbook §14.1 to §14.9)

R. Sekar
- If there a bijection $f: A \rightarrow B$ then $|A| = |B|

- More generally, if there is a mapping $g: A \rightarrow B$
  with “$=1$ arrow out” and “$=k$ arrow in” properties
  then $|B| = |A|/k$ (Division Rule)
Counting and Mappings

- If there a bijection \( f: A \rightarrow B \) then \( |A| = |B| \)

- More generally, if there is a mapping \( g: A \rightarrow B \) with “1 arrow out” and “\( =k \) arrow in” properties then \( |B| = |A|/k \) (Division Rule)

- Additionally, we use what we already know about sizes of sets:
  - If \( |A| = n \) then \( |P(A)| = 2^n \)
  - If \( A \) and \( B \) are disjoint, \( |A \cup B| = |A| + |B| \) (Sum Rule)
  - Size of \( A \times B \) is \( |A| \times |B| \) (Product Rule)
If there are 60 students enrolled in CSE 150 and 50 students enrolled in CSE 350, how many students are enrolled across the two courses?
If there are 60 students enrolled in CSE 150 and 50 students enrolled in CSE 350, how many students are enrolled across the two courses?

Since students don’t take the two courses simultaneously, the two sets are disjoint, so we can apply the sum rule.

The same rule cannot be applied to find the combined number of students across CSE 150 and AMS 210. They may be taken simultaneously, so the sets overlap.
World’s Best Donut Shop (WBDS) is so famous that everyone wants to buy a donut there.

To maximize the number of customers serviced, the shop limits each customer to just one Donut.

To ensure a unique customer experience, their Chef insists that every donut be distinct.

If WBDS offers a choice of
- 10 possible flavors
- 8 possible fillings
- 5 possible toppings
- 8 possible sprinkles

How many customers can WBDS serve in a day?
Product Rule

- The donut problem can be reduced to one of counting sequences
- A donut is characterized by the sequence \((F, I, T, S)\) representing the flavor, filling, topping and sprinkle choices

**Product Rule**

If \(P_1, P_2, \ldots, P_n\) are finite sets, then

\[ |P_1 \times P_2 \times \cdots \times P_n| = |P_1| \cdot |P_2| \cdot \cdots \cdot |P_n| \]

- Thus, the number of possible donuts = \(|F| \cdot |I| \cdot |T| \cdot |S| = 10 \times 8 \times 5 \times 8 = 3200\)

**Key Assumption:** The choices are independent of each other
How many ways are there to arrange a pawn, a knight and a rook on a chessboard such that no two pieces occupy the same row or column?

Let us represent these positions as the sequence \((r_p, c_p, r_k, c_k, r_r, c_r)\).

The pawn can be in any of the 8 rows and columns, so \(r_p\) and \(c_p\) have 8 possible values.
The knight can be in one of the remaining 7 rows/columns, so \(r_k\) and \(c_k\) have 7 possible values.
The rook can be in one of the 6 rows and columns that are free after placing the pawn and the knight, so \(r_r\) and \(c_r\) have 6 possible values.

The total number of positions is hence \(8 \cdot 8 \cdot 7 \cdot 7 \cdot 6 \cdot 6 = 112,896\).
Generalized Product Rule: Pieces on a Chessboard

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**Generalized Product Rule**

If there are \(n_i\) possible entries for the \(i\)th position in a length-\(k\) sequence, then the number of distinct sequences is \(n_1 \cdot n_2 \cdot \ldots \cdot n_k\).
Generalized Product Rule: Pieces on a Chessboard

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- The total number of positions is hence \(8 \cdot 8 \cdot 7 \cdot 7 \cdot 6 \cdot 6 = 112,896\)
A permutation of a set $S$ is a sequence that contains every element of $S$ exactly once.

How many distinct permutations of $S$ are there, if $|S| = n$?

- The first element of the sequence can be any of $n$ elements.
- Second element can be one of the remaining $n-1$ elements.
- Third element can be one of the remaining $n-2$ elements.
- Fourth element can be one of the remaining $n-3$ elements.
- Continuing on, we arrive at $n!$
Words of length $r$ over an alphabet of size $n$

- If letters can be repeated:
  - Every letter can be one of $n$ letters in the alphabet.
  - So, the total number of possibilities is $n^r$.

- If letters cannot be repeated:
  - Then we are asking for $nP_r$, $r$-length permutations of $n$ letters.
  - The first letter can be chosen in $n$ ways
  - The second letter can be chosen from the remaining $n - 1$ letters
  - The $k$’th letter can be chosen from $n - k + 1$ letters,
    - i.e., after leaving out the letters already used in the preceding $k - 1$ letters
  - So, the number is:
    $n \cdot (n - 1) \cdots (n - r + 1) = \frac{n!}{(n - r)!}$
Award Distribution

In how many ways can we distribute awards $A_1, \ldots, A_r$ to $n$ persons?

- We can represent the award as a sequence $(p_1, \ldots, p_r)$, where the $p_i$ denotes the person winning the award $A_i$.
- Thus, the number of possibilities is $n^r$

But what if each person can win only one award?

- For the $i$th award, the $i - 1$ persons that won awards 1 through $i - 1$ are not eligible

Using this reasoning, we can calculate the number of possibilities as:

$$n \cdot (n-1)(n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!} = \binom{n}{r}$$
Division Rule

How many ways are there to arrange two identical rooks on a chessboard such that they occupy distinct rows and columns?

- If the rooks are distinct, e.g., black and white, the answer is $8 \cdot 8 \cdot 7 \cdot 7 = 3136$
- What if the rooks are of the same color?
  - Note that the position $(r_1, c_1, r_2, c_2)$ is indistinguishable from $(r_2, c_2, r_1, c_1)$ because the two rooks are identical

By applying division rule, we arrive at the correct number for identical rooks:

\[
\frac{3136}{2} = 1568
\]
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**Division Rule**

If \(f : A \rightarrow B\) is a \(k\)-to-1 onto function then \(|A| = k \cdot |B|\)

(Such a function has \(\geq 1\) arrow out and \(\geq k\) arrow in properties.)
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Knights around a Circular Table

- How many distinct ways can \( n \) knights be seated around a circular table?
- Two seatings are considered the same if the two knights sitting next to each knight remains the same in both seatings.

Let there be \( n \) knights and \( m \) seats around a table. A seating can be captured by listing the seats in order \((k_1, k_2, \ldots, k_m)\), which translates to \( \frac{n!}{(n-m)!} \) seatings.

But many sequences represent the “same” seating arrangement. The first row lists the participants in clockwise order, starting from one of the \( m \) seats. The corresponding anti-clockwise order listing is shown in the second row.

\[
\begin{align*}
(k_1, k_2, \ldots, k_m) &\quad (k_2, \ldots, k_m, k_1) &\quad (k_3, \ldots, k_m, k_1, k_2) &\quad \cdots &\quad (k_m, k_1, \ldots, k_{m-1}) \\
(k_1, k_m, k_{m-1} \ldots, k_2) &\quad (k_2, k_1, k_m, \ldots k_3) &\quad (k_3, k_2, \ldots k_4) &\quad \cdots &\quad (k_m, k_{m-1}, k_{m-2}, \ldots, k_1)
\end{align*}
\]

Using division rule, we obtain the number of distinct seatings as

\[
\frac{n!}{(n-m)!} \cdot \frac{1}{2m} = \frac{n!}{2(n-m)!m}
\]
Combinations aka Counting Subsets

- How many distinct 5-card poker hands can be dealt from a 52-card deck?
- How many ways can I select 3 toppings for my pizza from the 10 available toppings?
Combinations aka Counting Subsets

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**Subset Rule**

A set of size $n$ has $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ distinct subsets of size $m$

- Start with a $m$-length permutations of the form $(i_1, i_2, \ldots, i_m)$
  - By the permutation rule, there are $\frac{n!}{(n-m)!}$ such sequences
- Apply division rule to collapse sequences corresponding to the same set:
  - There are $m!$ permutations of an $m$-element set
- Thus, there are $\frac{n!}{(n-m)!m!} = \binom{n}{m}$ distinct subsets (aka combinations)
Despite its unparalleled reputation, WBDS management is unhappy with the sales volume. They propose selling one box of dozen donuts per customer.

Chef’s agrees to sell the same donut to multiple customers, as long as no two customers get the exact same dozen.

How many donuts can they sell in a day?

Problem: How many distinct subsets of 12 can be drawn from 3200?

\[
\approx 2^{3200} \approx 2^{3200} \times 10^{33} \quad (\text{i.e., } 2^{3200} \times 10^{33})
\]

So, WBDS can sell up to \(12 \times 3200 \approx 28\) decillion donuts!

What is the minimum donuts per box to be able to sell to everyone in the world?

\[
\approx 5 \times 10^{11} \quad \text{will almost do!}
\]
Despite its unparalleled reputation, WBDS management is unhappy with the sales volume. They propose selling one box of dozen donuts per customer.

Chef’s agrees to sell the same donut to multiple customers, as long as no two customers get the exact some dozen.

How many donuts can they sell in a day?

Problem: How many distinct subsets of 12 can be drawn from 3200?

\[
\binom{3200}{12} \approx 2.36 \text{ Decillion}!!! \text{ (i.e., } 2.36 \times 10^{33}\text{)}
\]

So, WBDS can sell up to \(12 \cdot \binom{3200}{12} \approx 28\) decillion Donuts!
Donut Selection II

Despite its unparalleled reputation, WBDS management is unhappy with the sales volume. They propose selling one box of dozen donuts per customer.

Chef’s agrees to sell the same donut to multiple customers, as long as no two customers get the exact some dozen.

How many donuts can they sell in a day?

Problem: How many distinct subsets of 12 can be drawn from 3200?

\[
\binom{3200}{12} \approx 2.36 \text{ Decillion!!! (i.e., } 2.36 \times 10^{33})
\]

So, WBDS can sell up to \(12 \cdot \binom{3200}{12} \approx 28\) decillion Donuts!

What is the minimum donuts per box to be able to sell to everyone in the world?

\[\binom{3200}{3} \approx 5.5B\] will almost do!
In how many ways can we split a set of size $n$ into subsets of size $k_1, k_2, \ldots, k_m$?

Each split is a collection of sets $\{A_1, A_2, \ldots, A_m\}$ such that $|A_i| = k_i$ and $\sum_{i=1}^{m} k_i = n$.

Given a permutation of $n$ elements, we can treat the first $k_1$ elements in this sequence as $A_1$, the next $k_2$ elements as $A_2$ and so on.

However, the same elements of $A_i$ are represented using $k_i!$ distinct permutations.

So we apply division rule for $A_1, \ldots, A_m$: 

$$\frac{n!}{k_1! k_2! \cdots k_m!}$$
Sequences of Subsets

- In how many ways can we split a set of size \( n \) into subsets of size \( k_1, k_2, \ldots, k_m \)?
- Each split is a collection of sets \( \{A_1, A_2, \ldots, A_m\} \) such that \( |A_i| = k_i \) and \( \sum_{i=1}^{m} k_i = n \).
- Given a permutation of \( n \) elements, we can treat the first \( k_1 \) elements in this sequence as \( A_1 \), the next \( k_2 \) elements as \( A_2 \) and so on.
- However, the same elements of \( A_i \) are represented using \( k_i! \) distinct permutations.

So we apply division rule for \( A_1, \ldots, A_m \):

\[
\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}
\]

\( \binom{n}{m} \) is called a binomial coefficient; \( \binom{n}{k_1,k_2,\ldots,k_m} \) is called a multinomial coefficient.
Defective Dollar Bills

Let us call a dollar bill defective if any digit in its 8-digit serial number is repeated.

How many defective dollar bills are possible?

Often, it is easier to count the complement of a set:

\[
|A| = |\cup| - |A|,
\]

Here, the complement set is one that contains no repeated digit.

If no digit is repeated, the number of possible dollar bills is

\[
\frac{10!}{(10-8)!} = \frac{10!}{2!}.
\]

If repetitions are permitted, number of possible bills is

\[
10^8.
\]

So, number of defective bills is

\[
10^8 - \frac{10!}{2} = 98185600.
\]

Fraction of non-defective bills is

\[
\frac{10 \cdot 9 \cdot 8 \cdot \cdots \cdot 3}{10^8} = \frac{1}{81}.
\]

Over 49 of 50 possible dollar bills are defective!
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- How many defective dollar bills are possible?
- Often, it is easier to count the complement of a set:

**Complement Rule**

\[ |A| = |U| - |\overline{A}| \]

If no digit is repeated, the number of possible dollar bills is 
\[ \frac{10!}{(10 - 8)!} = \frac{10!}{2!} \]

If repetitions are permitted, the number of possible bills is 
\[ 10^8 \]

So, the number of defective bills is 
\[ 10^8 - \frac{10!}{2!} = 98185600. \]

Fraction of non-defective bills is 
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Defective Dollar Bills

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- Here, the complement set is one that contains *no* repeated digit.
- If no digit is repeated, the number of possible dollar bills is \(10!/(10 - 8)! = 10!/2\)
- If repetitions are permitted, number of possible bills is \(10^8\)

- So, number of defective bills is \(10^8 - 10!/2 = 98185600\).
Defective Dollar Bills

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\textbf{Complement Rule}

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Here, the complement set is one that contains \textit{no} repeated digit.

- If no digit is repeated, the number of possible dollar bills is \(10!/(10 - 8)! = 10!/2\)
- If repetitions are permitted, number of possible bills is \(10^8\)

So, number of defective bills is \(10^8 - 10!/2 = 98185600\).

Fraction of non-defective bills is \((10 \cdot 9 \cdot 8 \cdot \ldots \cdot 3)/10^8 = 1.81\%\)

Over 49 of 50 possible dollar bills are defective!
Complement Rule: Additional Examples

- If you toss a fair coin $n$ times, what is the likelihood there will be one or more heads?
  - Very easy to count sequences that contain no heads: there is exactly one!
  - It is also easy to total number of possible sequences: $2^n$
  - With a fair coin, all sequences are equally likely, so the probability is $(2^n - 1)/2^n$
Complement Rule: Additional Examples

- If you toss a fair coin \( n \) times, what is the likelihood there will be one or more heads?
  - Very easy to count sequences that contain no heads: there is exactly one!
  - It is also easy to total number of possible sequences: \( 2^n \)
  - With a fair coin, all sequences are equally likely, so the probability is \( \frac{2^n - 1}{2^n} \)

- In a group of five students, what is the likelihood of finding two students that were born on the same day of the week?
  - Easier to count instances where people were born on distinct days
    - \( \binom{7}{5} \) ways for five students to be born on 5 distinct days.
    - \( 7^5 \) ways if there are no constraints.
  - So, the desired probability, assuming that students are equally likely to be born any day of the week, is \( \frac{7^5 - \binom{7}{5}}{7^5} = 0.99875 \)
Summary of Basic Counting Rules

Product rule: \(|P_1 \times P_2 \times \cdots \times P_n| = |P_1| \cdot |P_2| \cdots \cdot |P_n|\) \quad \text{If choices are independent}

Sum rule: \(|P_1 \cup P_2 \cup \cdots \cup P_n| = |P_1| + |P_2| + \cdots + |P_n|\) \quad \text{If sets are disjoint}

Complement rule: \(|A| = |U| - |\overline{A}|\)

- Alternative form: If \(A \subseteq S\) then \(|A| = |S| - |S - A|\)

Division rule: If \(f : A \rightarrow B\) is a \(k\)-to-1 onto function then \(|A| = k \cdot |B|\).

Permutation rule: The number of \(r\)-length permutation of \(n\) elements is \(\frac{n!}{(n-r)!}\).

Combinations/subset rule: A set of size \(n\) has \(\binom{n}{m} = \frac{n!}{m!(n-m)!}\) distinct subsets of size \(m\).
How many distinct words can be formed by permuting letters in the word Bookkeeper?
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Consider all permutations, then use division rule to eliminate duplicates.
Words from Repeated Letters: Bookkeeper

- How many distinct words can be formed by permuting letters in the word Bookkeeper?
- Consider all permutations, then use division rule to eliminate duplicates
- Total permutations = 10!
- Repetitions: 3 E’s, 2 O’S, 2 K’s

\[
\frac{10!}{3!2!2!} = 151200
\]
How many 5-card hands have all four suits of some rank?
- $8\spadesuit 8\diamondsuit 8\heartsuit 8\clubsuit$
- $2\spadesuit Q\spadesuit 2\diamondsuit 2\heartsuit 2\clubsuit$
How many 5-card hands have all four suits of some rank?

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First pick the repeating rank: 1 of 13 ways
Next pick the fifth card: 1 of 48 ways
Total: $13 \cdot 48 = 624$

What is the probability of being dealt a four-of-a-kind hand?
Divide by number of possible 5-card hands

$$\frac{624}{52^5}, 598,960 = 0.024\%$$
How many 5-card hands have all four suits of some rank?

- $8\spadesuit 8\heartsuit 8\diamondsuit 3\diamondsuit$
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Poker Hands: Four of a Kind

- How many 5-card hands have all four suits of some rank?
  - 8♠ 8♦ 8♥ 8♣ 3♣
  - 2♠ Q♣ 2♦ 2♥ 2♣

- First pick the repeating rank: 1 of 13 ways
- Next pick the fifth card: 1 of 48 ways
- Total: $13 \cdot 48 = 624$

- What is the probability of being dealt a four-of-a-kind hand?
  - Divide by number of possible 5-card hands
    \[
    \frac{624}{\binom{52}{5}} = \frac{624}{2,598,960} = 0.024\% 
    \]
A *Full House* is a hand with three cards of one rank and two of another rank.

- Choose the rank of three cards: 13 ways
- Choose the suits of three cards: \(4^3 = 4\cdot 1^3 = 4\)
- Choose the rank of two cards: 12 ways
- Choose the suits of two cards: \(4^2 = 6\)

Apply the product rule to get

\[13 \cdot 4 \cdot 12 \cdot 6 = 3744\]

The probability is significantly higher than 4-of-a-kind, but still very low: 0.14%
Poker Hands: Full House

- A *Full House* is a hand with three cards of one rank and two of another rank.

- Choose the rank of three cards: 13 ways

- Choose the suits of three cards: \( \binom{4}{3} = \binom{4}{1} = 4 \)

- Choose the rank of two cards: 12 ways

- Choose the suits of two cards: \( \binom{4}{2} = 6 \)

- Apply the product rule to get \( 13 \cdot 4 \cdot 12 \cdot 6 = 3744 \)
Poker Hands: Full House

- A *Full House* is a hand with three cards of one rank and two of another rank.

- Choose the rank of three cards: 13 ways

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- Choose the rank of two cards: 12 ways

- Choose the suits of two cards: \( \binom{4}{2} = 6 \)

- Apply the product rule to get \(13 \cdot 4 \cdot 12 \cdot 6 = 3744\)

- The probability is significantly higher than 4-of-a-kind, but still very low: 0.14%
Poker Hands: Non-repeating Ranks

- Approach 1

Choose the 5 ranks in 13 \(\binom{13}{5}\) ways
Choose the suit of each card in 4 ways
Total: \(13 \binom{13}{5} \cdot 4^5 = 1,317,888\) ways

Approach 2

Count 5-card sequences with distinct ranks, then use division rule
Choose the first card in 52 ways, second card in 48 ways, and so on
Finally divide by 5! to get:
\(\frac{52 \cdot 48 \cdot 44 \cdot 40 \cdot 36}{5!} = 1,317,888\) ways

Dividing by the total number of hands, we get
\(\frac{1,317,888}{2,598,960} = \frac{5071}{42} = 0.1217\).
Poker Hands: Non-repeating Ranks

Approach 1
- Choose the 5 ranks in $\binom{13}{5}$ ways
- Choose the suit of each card in 4 ways
- Total: $\binom{13}{5} \cdot 4^5 = 1,317,888$ ways

Approach 2
- Count 5-card sequences with distinct ranks, then use division rule
- Choose the first card in 52 ways, second card in 48 ways, and so on
- Finally divide by 5! to get:
  $$\frac{52 \cdot 48 \cdot 44 \cdot 40 \cdot 36}{5!} = 1,317,888$$
- Dividing by the total number of hands, we get
  $$\frac{1,317,888}{52!} = 0.5071$$
Poker Hands: Non-repeating Ranks

- **Approach 1**
  - Choose the 5 ranks in \( \binom{13}{5} \) ways
  - Choose the suit of each card in 4 ways
  - Total: \( \binom{13}{5} \cdot 4^5 = 1,317,888 \) ways

- **Approach 2**
  - Count 5-card sequences with distinct ranks, then use division rule
    - Choose the first card in 52 ways, second card in 48 ways, and so on
    - Finally divide by 5! to get: \( \frac{52 \cdot 48 \cdot 44 \cdot 40 \cdot 36}{5!} = 1,317,888 \) ways

- Dividing by the total number of hands, we get

\[
\frac{1,317,888}{\binom{52}{5}} = \frac{1,317,888}{2,598,960} = 0.5071
\]
Poker Hands: Two Pairs

- Two cards of one rank, two cards of another rank, a fifth card of different rank

First pair can be chosen in \( \binom{13}{2} \) ways.

Second pair can be chosen in \( \binom{12}{2} \) ways.

Fifth card can be chosen in 44 ways (leave out all suits of the two chosen ranks).

Since we can't distinguish between the first and second pairs, divide by 2.

\[
\frac{\binom{13}{2} \cdot \binom{12}{2} \cdot 44}{2} = 12,355
\]
Poker Hands: Two Pairs

- Two cards of one rank, two cards of another rank, a fifth card of different rank
- First pair can be chosen in
  \[
  \binom{13}{1} \binom{4}{2} = 78 \text{ ways}
  \]
- Second pair can be chosen in
  \[
  \binom{12}{1} \binom{4}{2} = 72 \text{ ways}
  \]
- Fifth card can be chosen in 44 ways (leave out all suits of the two chosen ranks)
- Since we can’t distinguish between the first and second pairs, divide by 2!
  \[
  \frac{78 \cdot 72 \cdot 44}{2} = 123,552 \text{ ways}
  \]
Poker Hands: Two Pairs — Alternate Method

- Pick the rank of the two pairs: \( \binom{13}{2} \) ways

- Pick the suit of the lower-ranked pair: \( \binom{4}{2} \) ways

- Pick the suit of the higher-ranked pair: \( \binom{4}{2} \) ways

- Pick the rank of extra card: 11 ways

- Pick the suit of the extra card: 4 ways

Total:

\[
\left( \binom{13}{2} \right) \cdot \left( \binom{4}{2} \right)^2 \cdot 11 \cdot 4 = 123,552 \text{ ways}
\]
How many hands contain at least one card from every suit?

Key point: the choice of the fifth card is not entirely independent of the first four.

For the first four cards, the suites are fixed, so there is a choice only for the ranks:

$$13 \times 13 \times 13 \times 13 = 13^4$$

The last card can be any of the remaining 48:

$$\frac{48}{2} = 24$$

So we need to divide in the end by 2:

$$\frac{13^4 \times 48}{2} = 685,464$$
Poker Hands: Every Suit

- How many hands contain at least one card from every suit?
- Key point: the choice of the fifth card is not entirely independent of the first four.
Poker Hands: Every Suit

How many hands contain at least one card from every suit?

Key point: the choice of the fifth card is not entirely independent of the first four.

For the first four cards, the suites are fixed, so there is a choice only for the ranks:

\[13 \cdot 13 \cdot 13 \cdot 13 = 13^4\]

The last card can be any of the remaining 48.

But if we switch the last card with the other card that has the same suit, we would have counted that as a separate hand.

So we need to divide in the end by 2.

\[
\frac{13^4 \cdot 48}{2} = 685,464
\]
Poker Hands: At least 4 Ranks

- Break into two cases:
  - Exactly 5 ranks (already done): 1,317,888 ways
  - Exactly 4 ranks

Choose the 4 ranks in $\binom{13}{4}$ ways
Choose which rank to repeat: 4 ways
Choose the suit of 3 non-repeating ranks: $4^3$ ways
Choose the suits of repeating rank in $4^2$ ways
Total: $13 \cdot 4 \cdot 4^3 \cdot 4^2 = 1,098,240$ ways
Total across two cases: 1,317,888 + 1,098,240 = 2,416,128 ways
Poker Hands: At least 4 Ranks

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    - Choose the suits of repeating rank in $\binom{4}{2}$ ways
    - Total: $\binom{13}{4} \cdot 4 \cdot 4^3 \cdot \binom{4}{2} = 1,098,240$ ways

- Total across two cases: $1,317,888 + 1,098,240 = 2,416,128$ ways
How many binary relations from $X$ to $Y$ are there? State your answer in terms of the cardinalities of $X$ and $Y$. 
Number of Bijections

Let $|X| = n$. How many bijections are there from $X$ to $X$?
How many $n$-bit sequences have exactly $k$ ones?
How many $n$-bit sequences have exactly $k$ ones?

You need to select a subset of $k$ positions that will be occupied by 1’s, while all other positions will be zeroes.

So, the number is

\[ \binom{n}{k} \]
To further increase profits, WBDS management wants to streamline production by cutting down the number of distinct donut types to 10.

The Chef continues to insist on a unique box of dozen for each customer.

How many customers can WBDS serve per day with these new rules?

You can map it to a bit strings problem! The 0’s represent the donuts, 1’s represent the boundary between donut types. We need 9 boundaries for 10 donut types, and 12 zeroes for 12 donuts. Thus, the total number of boxes is

\[ \approx 300 \]

How about the general case of a box of \( n \) donuts drawn from \( m \) types? We need \( m - 1 \) ones and \( n \) zeroes, so the number is

\[ n + \frac{m - 1}{m - 1} = n + \frac{m - 1}{m - 1} \]

57 / 99
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Thus, the total number of boxes is \( \binom{21}{9} \approx 300K \)
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How about the general case of a box of \( n \) Donuts drawn from \( m \) types?

- We need \( m - 1 \) ones and \( n \) zeroes, so the number is \( \binom{n+m-1}{m-1} = \binom{n+m-1}{n} \)
What if you add a requirement that there be at least one donut of each type?
Donut Selection: Variants

- What if you add a requirement that there be at least one donut of each type?
- Put one donut of each type into the box, then count the ways to choose the other two
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**Approach 1: Map to binary strings with 2 zeroes and 9 ones**
- \( \binom{11}{9} = \binom{11}{2} = \frac{11 \cdot 10}{2} = 55 \)
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- \( \binom{11}{9} = \binom{11}{2} = \frac{11 \cdot 10}{2} = 55 \)

Approach 2: Decompose into union of disjoint sets
- The last two donuts can have (i) the same type, or (ii) distinct types
  - There are 10 choices for (i), and
  - \( \binom{10}{2} = \frac{10 \cdot 9}{2} = 45 \) choices for (ii)
- The total is again 55.
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- The total is again 55.

Often, there are multiple ways of counting
- but all should yield the same answer: use this to check your approach.
Choosing Books

- There are 6 books on a shelf. You want to choose 3 books such that no consecutive books are chosen.
Choosing Books

- There are 6 books on a shelf. You want to choose 3 books such that no consecutive books are chosen.
- What if there are 25 books on the shelf, and you want to choose 10?
Choosing Books

There are 6 books on a shelf. You want to choose 3 books such that no consecutive books are chosen.

What if there are 25 books on the shelf, and you want to choose 10?

Selection problems like this can often be mapped to bit strings:

- Use 25 bits to represent 25 books, with 1’s for selected books and 0’s for unselected ones.
Choosing Books

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What if there are 25 books on the shelf, and you want to choose 10?

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How to incorporate the constraints?

- Select 10 books: There should be exactly ten 1-bits
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  - We are done with all selections, so we are not going to select any more books
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For the bit string, this constraint becomes:

- “Ten 1-bits and fifteen zero bits with all but the last 1 to be followed by a 0”
Choosing Books (Continued)

- Consider “Ten 1-bits and fifteen zero bits with all but the last 1 to be followed by a 0”
- The constraint about 0-bit following a 1-bit is precisely captured by “taping” a zero to all but the last 1.
- Since there are nine such 1’s, this effectively reduces the number of positions by 9
Choosing Books (Continued)

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- Thus, we need to count bit strings with $25 - 9 = 16$ positions with ten 1’s.
  - So, the number of possible selections is $\binom{16}{10} = 8008$
Choosing Books (Continued)

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- Thus, we need to count bit strings with $25 - 9 = 16$ positions with ten 1’s.
  - So, the number of possible selections is $\binom{16}{10} = 8008$
- Can we generalize to $m$ books out of $n$?
  - The “taping” step above reduces the number of positions from $n$ to $n - m + 1$
  - So, the number is $\binom{n-m+1}{m}$
Partitioning an Integer

Let $S_{n,k}$ be the possible non-negative integer solutions to the inequality

$$x_1 + x_2 + \cdots + x_k = n$$

How many solutions are there?
Partitioning an Integer

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$$x_1 + x_2 + \cdots + x_k = n$$

How many solutions are there?

This is just the Donut problem!

- Make a box of $n$ Donuts, drawing from $k$ different types.
- So, the solution is

$$S_{n,k} = \binom{n + k - 1}{n}$$
Let \( I_{n,k} \) be the possible non-negative integer solutions to the inequality
\[
x_1 + x_2 + \cdots + x_k \leq n
\]

How many solutions are there?
Partitioning an Integer

Let $I_{n,k}$ be the possible non-negative integer solutions to the inequality

$$x_1 + x_2 + \cdots + x_k \leq n$$

How many solutions are there?

**Can be reduced the equality case from previous slide:**

$$x_1 + x_2 + \cdots + x_k + x_{k+1} = n$$

For any value $n' \leq n$ such that $x_1 + x_2 + \cdots + x_k = n'$, $x_{k+1} = n - n'$

- Thus there is a bijection between the equality and inequality formulations.
- So, the solution is

$$I_{n,k} = \binom{n+k}{n}$$
Let $L_{n,k}$ be the length of $k$ weakly increasing sequences of non-negative integers, i.e.,

$$y_1 \leq y_2 \leq \cdots \leq y_k \leq n$$

How many such sequences are there?

This is the same problem from the last slide!

Let $y_i$ denote the sum of $x_1$ through $x_i$

$$y_i = \sum_{j=1}^{i} x_j$$
A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?
Pigeonhole Principle

- A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?

**Simple Version**

*If there are more pigeons than the holes they occupy, then there must be at least two pigeons in some hole*
Pigeonhole Principle

A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?

**Simple Version**

If there are more pigeons than the holes they occupy, then there must be at least two pigeons in some hole.

**Formal Version**

If $|A| > |B|$ then every total function $f: A \rightarrow B$ maps at least two different elements of $A$ to an element of $B$. 
Hairs on Heads

How many people riding the NYC subway in a day have the same number of hairs on their heads?
Hairs on Heads

- How many people riding the NYC subway in a day have the same number of hairs on their heads?
- Let us say that each head has a maximum of 100K hairs
- NYC daily subway (pre-pandemic) ridership is about 4.3M
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- Let us say that each head has a maximum of 100K hairs
- NYC daily subway (pre-pandemic) ridership is about 4.3M

**Generalized Pigeonhole Principle**

If $|A| > k \cdot |B|$ then every total function $f: A \to B$ maps at least $k + 1$ different elements of $A$ to an element of $B$
Subset With Same Sum

- Suppose that we generate 100 random numbers with 25 digits:
  0020480135385502964448038
  5763257331083479647409398
  0489445991866915676240992
  ...

- Will there be two subsets of these 25-digit numbers that add up to the same value?
Union of Overlapping Sets: Inclusion-Exclusion Principle

- So far, we studied union of disjoint sets, where:

\[ |S_1 \cup S_2| = |S_1| + |S_2| \]

- What happens when the sets overlap?
Union of Overlapping Sets: Inclusion-Exclusion Principle

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  \[ |S_1 \cup S_2| = |S_1| + |S_2| \]

- What happens when the sets overlap?

**Union of Two Sets**

\[ |S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2| \]
Union of Three Sets

\[ S_1 \cup S_2 \cup S_3 = |S_1| + |S_2| + |S_3| - (|S_1 \cap S_2| + |S_2 \cap S_3| + |S_3 \cap S_1|) + |S_1 \cap S_2 \cap S_3| \]
Union of Three Sets

\[ |S_1 \cup S_2 \cup S_3| = (|S_1| + |S_2| + |S_3|) - (|S_1 \cap S_2| + |S_2 \cap S_3| + |S_3 \cap S_1|) + |S_1 \cap S_2 \cap S_3| \]
Union of $n$ sets

### Inclusion-Exclusion for $n$ Sets

$$\left| \bigcup_{i=1}^{n} S_i \right| = \sum_{i=1}^{n} |S_i| - \sum_{1 \leq i < j \leq n} |S_i \cap S_j| + \sum_{1 \leq i < j < k \leq n} |S_i \cap S_j \cap S_k| \cdots (-1)^{n-1} \left| \bigcap_{i=1}^{n} S_i \right|$$

Or, more compactly:

$$\left| \bigcup_{i=1}^{n} S_i \right| = \sum_{l \in \varnothing \cup \{1, 2, \ldots, n\}} (-1)^{|l|+1} \left| \bigcap_{i \in l} S_i \right|$$
Sequences with 42, 04 or 60

How many permutations of \{0, 1, 2, \ldots, 9\} contain a 42, 04 or 60?

- Permutations containing 42: “Fuse” 4 and 2 together, treat as if they are a single symbol. Now we have 9 symbols, for 9!
- Permutations containing 04 and 60 are also 9!
- Permutations containing two of these pairs would be 8!
- Permutations containing all three would be 7!

Using inclusion-exclusion principle, we arrive at:

\[3 \cdot 9! - 3 \cdot 8! + 7!\]
Sequences with 42, 04 or 60

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- Permutations containing two of these pairs would be \(8!\)

- Permutations containing all three would be \(7!\)

- Using inclusion-exclusion principle, we arrive at
  \[3 \cdot 9! - 3 \cdot 8! + 7!\]

- How about sequences with 42, 62, or 60?
How many numbers between 1 and 100 contain a 5 in them?
Numbers Containing Certain Digits

- How many numbers between 1 and 100 contain a 5 in them?
- How many numbers between 1 and 1000 contain a 5 in them?
How many numbers between 1 and 100 contain a 5 in them?

How many numbers between 1 and 1000 contain a 5 in them?

How many numbers between 1 and 100 contain a 5 in them?
Numbers Containing Certain Digits

- How many numbers between 1 and 100 contain a 5 in them?
- How many numbers between 1 and 1000 contain a 5 in them?
- How many numbers between 1 and 1B contain a 5 in them?
- *Inclusion-exclusion can be cumbersome to use.* You should first check if other rules (e.g., complement) are applicable.
Summary of Counting Rules

Product rule: \(|P_1 \times P_2 \times \cdots \times P_n| = |P_1| \cdot |P_2| \cdots |P_n|\) \hspace{1cm} \text{If choices are independent}

Sum rule: \(|P_1 \cup P_2 \cup \cdots \cup P_n| = |P_1| + |P_2| + \cdots + |P_n|\) \hspace{1cm} \text{If sets are disjoint}

Complement rule: \(|A| = |U| - |\overline{A}|\) \hspace{1cm} \text{(Alternative form: If } A \subseteq S \text{ then } |A| = |S| - |S - A|\)

Division rule: If \(f: A \longrightarrow B\) is a \(k\)-to-1 onto function then \(|A| = k \cdot |B|\).

Permutation rule: The number of \(r\)-length permutation of \(n\) elements is \(\frac{n!}{(n-r)!}\).

Combinations/subset rule: A set of size \(n\) has \(\binom{n}{m} = \frac{n!}{m!(n-m)!}\) distinct subsets of size \(m\).

Bijection with bit strings.

Pigeon Hole Principle.

Inclusion-Exclusion Principle.